Finite element methods for 3D eddy current problems in bounded domains subject to realistic boundary conditions. An application to metallurgical electrodes.

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Summary
This paper deals with the finite element solution of the eddy current problem in bounded domains. The starting point is an overview of the most used finite element techniques to approximate eddy current problems. The focus of the article is on the analysis of weak formulations based on the magnetic or the electric field, approximated by means of edge elements. The most outstanding characteristic is the treatment of boundary conditions feasible from the point of view of applications. Theoretical results concerning the continuous weak formulations and error estimates for the discrete problems are reported. Applications of the proposed techniques to electrodes of metallurgical electric furnaces are discussed and numerical results are given.

1 INTRODUCTION
Numerical simulation plays an important role in electrical engineering to optimize the design and operation conditions of electromagnetic devices such as electrical machines, induction heating systems, transformers, waveguides, microwaves, etc. The behavior of these devices is governed by Maxwell equations, which cannot be solved, in general, by using analytical methods. This is why the mathematical and numerical analysis of Maxwell equations has experimented during the last thirty years an important development in different areas of applied mathematics and engineering. We refer the reader to the books by Bossavit [38], Chari and Silvester [50], Hameyer and Belmans [65], Jin [75], Monk [93], Silvester and Ferrari [122], and Sykulski [123], just to list a representative sampling of text books devoted to numerical solution of electromagnetic problems.

Among the numerical methods found in the literature to approximate Maxwell equations, the finite element method is the most extended. Its main advantages are its geometric flexibility and the richness in theoretical mathematical tools useful to analyze the approximation of the problem. We notice, however, that Maxwell equations concern the whole space; so it is necessary to define suitable boundary conditions in order to use the finite
element method. Because of this, we can also find an important number of papers in the literature which couple the finite element method with the boundary element method (BEM-FEM methods) combining the advantages and disadvantages of each of them (see for instance [37, 39, 42, 43, 66, 87, 88, 89]).

The finite element method was introduced in electrical engineering calculations in 1970 and, since then, it has been applied to the simulation of a great variety of electromagnetic problems in static and transient state in two and three dimensions: electrostatics ([49, 59, 81]), magnetostatics ([59, 60, 80, 81, 109, 110, 114]), eddy current ([1, 2, 3, 6, 7, 8, 9, 19, 20, 21, 23, 24, 25, 26, 28, 29, 30, 33, 35, 62, 78, 79, 84, 94, 95, 96, 98, 101, 113, 115, 116, 117, 118, 121, 125, 126, 127]), wave propagation phenomena ([18, 106, 107]), etc. Nowadays, it is the basis of several commercial codes such as Ansys, Femlab, Flux, Magnet, MSC/Emas, Opera, etc., and the computational advances in this matter have been very important. We refer the reader to [123] for a description of most of these codes and further references. The development of different finite element methods and mathematical tools to analyze and numerically solve Maxwell equations has been one of the main directions of research during the past decades. From a mathematical point of view, the problem involves different difficulties depending on the simplified assumptions used in the full system of Maxwell equations, that is, equations in stationary state, transient state, quasi-static, linear, non-linear, etc. In particular, an important number of recent publications are devoted to approximate the low-frequency Maxwell equations in harmonic regime, namely, the so-called eddy current model.

The eddy current problem is obtained from Maxwell equations by assuming that all fields are harmonic and the frequency is low enough as to neglect the electric displacement in Ampere’s Law. Such a situation happens, for instance, in problems related to electric machines working at power frequencies (see for instance [9, 38, 75]). The operation of many electromagnetic devices depends on the circulation of eddy currents in their conducting parts and this is why the numerical solution of this problem has become an important research area during the last decades. Sometimes, the geometry of the device and the presence of symmetric operational conditions, allow solving a two-dimensional problem (plane or axisymmetric), which leads to important savings in computational effort (see for instance [13, 14, 16, 47, 48, 104, 113, 121]. However, a full three-dimensional analysis is needed in many real engineering problems. For instance, the investigation of the authors in eddy current problems has been motivated by the need of three-dimensional thermoelectrical simulation of electrodes in metallurgical electric furnaces for silicon and ferrosilicon production.

From the mathematical point of view, thermoelectrical modeling of electrodes leads to coupling the eddy current problem with the heat transfer equation. This coupling is often present in many other industrial problems and its mathematical analysis can be very complex; recent publications concerning this subject in two and three-dimensional domains are [17, 41, 51, 52, 53, 104, 105, 108] and there are still many open questions. However, the analysis of the coupled problem starts with the investigation of the three-dimensional eddy current model, which represents itself an enough complex problem and is the basis to analyze other coupled phenomena like those arising in magnetohydrodynamics or electromechanics.

An important number of formulations and finite element methods to solve the eddy current problem in three-dimensional bounded domains can be found in the literature. The main difference lies in the choice of the primary unknown. There is a group of papers devoted to solve the problem in terms of certain scalar and vector potentials ([2, 3, 26, 28, 29, 33, 83, 90, 91, 92, 94, 95, 96, 117]) and another group using formulations in terms of the magnetic field ([5, 6, 8, 19, 20, 23, 24, 79, 124]) or the electric field ([4, 21, 22, 62]). Concerning the development of finite element methods, those based on piecewise linear functions were the first to emerge to solve the potential formulations. However, since
Nédélec [102] introduced the edge finite elements in 1980, their presence in eddy current problems increased notably and, nowadays, they seem to be the most attractive option as they allow overcoming some drawbacks present in piecewise linear elements. We point out that parallel to the development of finite element methods, there has been an important progress in the mathematical analysis of the eddy current problem. Although some of the suggested formulations have not been fully studied yet, those based on edge elements has been the subject of an extensive research in the last years. In particular, the mathematical justification of weak formulations and convergence properties of the discrete problem have been the main topics.

We notice that the eddy current problem in bounded domains involves two important difficulties, which, in fact, are closely related to industrial applications: the topology of the domain and the boundary conditions. In most practical situations, it is necessary to solve the electromagnetic problem in a domain which contains conducting materials and non-conducting ones (dielectrics), the equations in these two parts being typically of different kind. Moreover, the treatment of multiply connected conductors or dielectrics in three-dimensional domains present special difficulties. As a consequence, the choice of the unknown in each subdomain is a crucial point to analyze the problem in domains with a general topology.

On the other hand, one of the main difficulties to study the eddy current problem in a bounded domain is to define adequate boundary conditions. These conditions must be mathematically suitable for the problem to be well posed, but at the same time physically realistic in the sense of involving only data actually attainable in practical scenarios. In fact, this point is sometimes forgotten in papers which analyze the problem from a purely mathematical point of view, in that they use boundary conditions which in general are not related to standard data usually accessible in industrial applications.

In this work, we will deal with these issues from the modeling, mathematical, and numerical points of view. Moreover, we will try to link the industrial problem with the mathematical one. The emphasis will be on studying finite element methods in three-dimensional bounded domains of general topology by using realistic boundary conditions.

The paper is structured as follows: In Section 2 we introduce the eddy current problem and review some of the solution techniques available in the literature, aiming to explain the evolution of the finite element method in this area. In Section 3, we consider different formulations of the problem. In each case, we describe the model and introduce finite element methods to solve the eddy current problem by using boundary conditions appropriate from the point of view of applications. We report the existence of solution of the continuous problems and convergence properties of the discrete ones. The techniques described in this section are suitable for numerical simulation of metallurgical electrodes, but they can also be applied to simulate other electrical devices working at power frequencies. In Section 4, we focus our attention on the mathematical modeling of metallurgical electrodes; we give a deep description of the physical problem, review the bibliography on this topic, and show some of the numerical results obtained with the proposed models. We end the paper drawing some conclusions and describing briefly some open lines of research.
2 THE EDDY CURRENT MODEL

2.1 Statement of the Problem

We start recalling the *Maxwell equations* governing electromagnetism ([76]):

\[
\begin{align*}
\frac{\partial \mathbf{D}}{\partial t} - \text{curl} \, \mathbf{H} &= -\mathbf{J}, \\
\frac{\partial \mathbf{B}}{\partial t} + \text{curl} \, \mathbf{E} &= 0, \\
\text{div} \, \mathbf{B} &= 0, \\
\text{div} \, \mathbf{D} &= q.
\end{align*}
\]

(1) (2) (3) (4)

Notations are standard:
- \( \mathbf{D} \) is the electric displacement,
- \( \mathbf{E} \) is the electric field,
- \( \mathbf{B} \) is the magnetic induction,
- \( \mathbf{H} \) is the magnetic field,
- \( \mathbf{J} \) is the current density, and
- \( q \) is the electric charge density.

Throughout the paper we will use boldface letters to denote vector fields and variables, as well as vector-valued operators. All the fields in the previous equations are functions which depend on the position \( \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \) and the time \( t \geq 0 \). Moreover, given a vector field

\[ \mathbf{F}(\mathbf{x}, t) = (F_1(\mathbf{x}, t), F_2(\mathbf{x}, t), F_3(\mathbf{x}, t)), \]

“div” and “curl” represent the *divergence* and *rotational* operators defined in Cartesian coordinates by

\[
\text{div} \, \mathbf{F} := \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3},
\]

\[
\text{curl} \, \mathbf{F} := \left( \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right).
\]

We will also use the “gradient operator” of scalar functions \( f(\mathbf{x}, t) \), which we denote “\( \text{grad} \)”: \( \text{grad} \, f := \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) \).

To obtain a closed system from Maxwell equations, we need more information which is provided by the *constitutive laws* between the above fields:

\[ \mathbf{B} = \mu \mathbf{H}, \]

\[ \mathbf{D} = \varepsilon \mathbf{E}, \]

and *Ohm’s law*

\[ \mathbf{J} = \sigma \mathbf{E}, \]

where
- $\mu$ is the magnetic permeability,
- $\epsilon$ is the electric permittivity, and
- $\sigma$ is the electric conductivity.

We consider only linear and isotropic materials, in which case $\epsilon$, $\mu$, and $\sigma$ are bounded scalar functions of the space variable $x$. Moreover, $\epsilon$ and $\mu$ are strictly positive, while $\sigma$ is strictly positive in conductors and vanishes in dielectrics.

We assume that the physical quantities vary sinusoidally as a function of time; that is, all fields have the following form:

$$F(x,t) = \text{Re} \left[ e^{i\omega t} F(x) \right],$$

where $F$ is a complex field (the so-called complex amplitude of $F$) and $\omega = 2\pi f$ is the angular frequency, with $f$ being the current frequency. In particular, this is a suitable assumption for the steady state of electric devices using alternating current.

In this case, Maxwell equations reduce to the so-called time-harmonic Maxwell equations, which are written, in terms of the complex amplitudes of the different fields, as follows:

$$\begin{align*}
i\omega D - \text{curl} \, H &= -J, \\
i\omega \mu H + \text{curl} \, E &= 0, \\
\text{div} \, B &= 0, \\
\text{div} \, D &= q,
\end{align*}$$

with $B = \mu H$, $D = \epsilon E$, and $J = \sigma E$.

Moreover, in many physical applications, it is possible to neglect the term involving the electric displacement in the Maxwell-Ampère law (1). This quasistatic assumption, also called eddy-current assumption, is reasonable when the displacement current $\frac{\partial D}{\partial t}$ is negligible relative to the other two terms in (1). In particular, in the harmonic regime, this assumption is valid if the current frequency $\omega$ is low enough. For a discussion of parameter ranges in which the quasistatic model is valid see for instance the book by Bossavit [38] or the paper by Ammari et al. [9] where the approximation is justified. By using the quasistatic assumption we obtain the time-harmonic eddy current model:

$$\begin{align*}
\text{curl} \, H &= J, \\
i\omega \mu H + \text{curl} \, E &= 0, \\
\text{div} \, B &= 0, \\
\text{div} \, D &= q,
\end{align*}$$

with $B = \mu H$, $D = \epsilon E$, and $J = \sigma E$.

By applying the div operator in equation (5), we also obtain the continuity equation for current density,

$$\text{div} \, J = 0.$$  

These equations are defined in the whole space but, in order to solve the problem with finite element methods, we need to consider a bounded domain and suitable boundary conditions, which will be introduced below.

In general, the computational domain of an eddy current problem includes conducting materials and the air around. Thus, in this section, we consider a bounded domain $\Omega \in \mathbb{R}^3$. 

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FEM for 3D eddy current problems in bounded domains subject to realistic boundary conditions.

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which consists of two parts, $\Omega_c$ and $\Omega_d$, occupied by conductors and dielectrics, respectively. Recall that, as stated above, the electric conductivity $\sigma$ vanishes in the dielectric domain.

Many formulations and finite element techniques to solve the problem (5)–(8) in three-dimensional bounded domains can be found in the literature. However, there is not a universal numerical method suitable to simulate all physical situations. In particular, we distinguish two groups of finite element methods: those based on scalar and vector potentials and those based on electric or magnetic fields.

### 2.2 Formulations Based on Scalar and Vector Potentials

In this section, we briefly recall the potential fields used to deal with the eddy current problems in three-dimensional domains. There are two main possibilities which combine electric and magnetic potentials: the so-called formulations $A^V$ and $T$.

#### 2.2.1 Formulation $A^V$ – $A$

Since $\text{div } B = 0$, we have that $B$ derives from a magnetic vector potential $A$, i.e.,

$$B = \text{curl } A. \quad (10)$$

On the other hand, since

$$\text{curl } E = -i\omega B = -i\omega \text{curl } A,$$

we obtain that $\text{curl}(E + i\omega A) = 0$. Then there exists a scalar potential $V$ such that

$$E + i\omega A = -\text{grad } V.$$

Therefore, the magnetic and electric fields can be written in terms of these potentials as follows:

$$E = -i\omega A - \text{grad } V,$$

$$H = \frac{1}{\mu} \text{curl } A.$$

By replacing these expressions in equation (5) and using Ohm’s law, $J = \sigma E$, we obtain

$$\text{curl } \left( \frac{1}{\mu} \text{curl } A \right) + i\omega \sigma A + \sigma \text{grad } V = 0 \quad \text{in } \Omega_c,$$

$$\text{curl } \left( \frac{1}{\mu} \text{curl } A \right) = 0 \quad \text{in } \Omega_d.$$ 

Notice that the definition (10) of the magnetic vector potential guarantees that $B$ is divergence free, so equation (7) is not necessary.

Thus, the $A^V$ formulation involves the magnetic vector potential and the scalar electric potential in the conductors, and only the magnetic vector potential in the dielectric domain.

#### 2.2.2 Formulation $T$ – $\phi$

Equation (9) leads to $J = \text{curl } T$, where $T$ is called the electric vector potential. On the other hand, since $\text{curl } H = \text{curl } T$ in conducting materials and $\text{curl } H = 0$ in dielectrics, we have

$$H = T - \text{grad } \phi \quad \text{in } \Omega_c,$$

$$H = -\text{grad } \phi \quad \text{in } \Omega_d.$$
where $\phi$ is called the scalar magnetic potential. We refer the reader to [26, 29, 100, 115] for further discussion concerning topological issues.

By replacing these expressions in (6) and (7), and using that $J = \sigma E$ and $B = \mu H$, we obtain

$$\text{curl} \left( \frac{1}{\sigma} \text{curl} \mathbf{T} \right) + i \omega \mu \mathbf{T} - i \omega \mu \text{grad} \phi = 0 \quad \text{in } \Omega_C,$$

$$\text{div} (\mu \mathbf{T} - \mu \text{grad} \phi) = 0 \quad \text{in } \Omega_C,$$

$$- \text{div} (\mu \text{grad} \phi) = 0 \quad \text{in } \Omega_D.$$

Then the $\mathbf{T} - \phi$ formulation involves the electric vector potential and the scalar magnetic vector potential in $\Omega_C$, and only the scalar magnetic potential in $\Omega_D$.

### 2.2.3 Numerical solution

Notice that the equations $B = \text{curl} \mathbf{A}$ and $J = \text{curl} \mathbf{T}$ specify the curl of the magnetic and electric vector potentials, respectively. However, Helmholtz theorem states that a vector field is uniquely determined only if both its curl and divergence are specified. The divergence of these fields may be defined freely without affecting the physical problem; this choice is called a gauge condition. In fact, the treatment of gauge conditions represents one of the main difficulties to solve the problem. Actually, in bounded domains, introducing a gauge condition will provide unique solution of the previous problems only if appropriate boundary conditions concerning $\mathbf{A}$, $V$, $\mathbf{T}$, and $\phi$ are also included (see for instance [26, 28]).

The eddy current model in terms of $\mathbf{A}$ and $V$ as well as its solution by using finite elements have been extensively studied by authors like Albanese and Rubinacci [1], Biddecombe et al. [25], Biro and coauthors [26, 27, 28, 29, 33], Morisue [94, 95, 96], Nakata et al. [98], and Renhart et al. [117]. The introduction of appropriate boundary conditions and gauging as well as the uniqueness of the potentials are the main problems discussed in these references.

On the other hand, finite elements methods to solve eddy current problems in terms of $\mathbf{T}$ and $\phi$ were studied by Nakata et al. [101], Ren [115], Renhart et al. [117], and also by Biro and coauthors [26, 29, 31].

The first finite element methods developed to solve these potential formulations consisted of linear nodal elements with three unknowns at each node to approximate the vector fields. However, the use of nodal finite elements leads to different difficulties, the analysis of which is beyond the scope of the present paper. In particular, these elements are not recommended for domains which contain materials with different permeabilities, or geometrical domains with reentrant corners in their outer boundary (see for instance [26, 75, 114] for further details). Some of these problems can be avoided by using the so-called edge elements, introduced by Nédélec [102]. Indeed, the use of edge elements to approximate vector potentials has become an important research area during the last years ([3, 26, 30, 35, 79, 80, 86, 97, 116]).

We notice again that the uniqueness of a vector potential needs of a gauge condition. Different techniques to impose it by using edge elements can be found in the literature, specially in the context of magnetostatics ([63, 79, 80, 86]). In fact, the choice of gauge largely affects the convergence of iterative methods used to solve the discretized linear systems (see for instance [69, 116] and references therein). Nevertheless, the gauge condition is not indispensable. Indeed, since we are interested in the curl of the potential, it is enough that the linear system provides a solution for the potential. This has been studied in different papers for magnetostatics and eddy current problems. In particular, it has been proved that the discretized system can be solved by using an appropriate conjugate gradient
method. In fact, some authors point out that a better convergence can be attained without imposing any explicit gauge condition ([26, 34, 35, 114, 116]).

Actually, the development of finite element methods based on potential formulations was very fast, and most of the commercial codes which allow solving eddy current problems include these techniques. In general these codes use nodal elements; only in the last years some commercial packages including edge elements have appeared.

We notice that the \( T - \phi \) model is in principle less expensive in terms of computational time and memory requirements than the \( A V - A \) model. However, the latter allows working with multiply connected domains in an easier way. Indeed, several publications are devoted to compare by means of numerical experiments the efficiency and accuracy of different potential formulations ([26, 29, 99, 100, 117]).

Concerning the mathematical analysis of potential formulations, many of them have not been fully studied yet. We quote the papers by Kanayama and Kikuchi [79] for the \( A V - A \) formulation and those by Maday et al. [84, 85] for the \( T - \phi \) model. Recently, Alonso Rodríguez et al. [6] have also analyzed some properties concerning the \( A V - A \) formulation.

We point out that other potential fields appearing in eddy current problems are the modified vector potential and the reduced vector potential (see for instance [26], for further details).

\section*{2.3 Formulations in Terms of the Magnetic or the Electric Field.}

The potential formulations introduced in the previous section present some disadvantages, such as the need of gauge conditions or the need of differentiating the computed potentials to recover the magnetic field. This is why finite element methods directly formulated in terms of the magnetic or the electric field started to emerge, since edge elements were introduced in computational electromagnetism.

We advance that the main advantage of edge elements is that they only enforce continuity of the tangential component and, therefore, they are suitable to represent the magnetic or the electric field, which in general present discontinuities at interfaces of different materials. Thus, edge elements do not impose over continuity as it happens when vector nodal elements are used. Another important feature of the electric or magnetic field formulation is that they do not need of any gauge condition.

The electric field formulation is obtained by eliminating the magnetic field in equations (5)--(8). Basically, from equation (6) we have

\[
H = -\frac{1}{i\omega\mu} \text{curl} \, E,
\]

and replacing this expression in (5) we obtain

\[
\text{curl} \left( \frac{1}{i\omega\mu} \text{curl} \, E \right) + \sigma E = 0.
\]

In a similar way, the magnetic field formulation is obtained by eliminating the electric field. From equation (5) and Ohm’s law, we have in conducting domains

\[
E = \frac{1}{\sigma} \text{curl} \, H,
\]

and replacing this expression in equation (6) we obtain

\[
\text{curl} \left( \frac{1}{\sigma} \text{curl} \, H \right) + i\omega\mu \epsilon H = 0,
\]
which is only valid in conductors. In dielectrics \( \text{curl} \mathbf{H} = 0 \) and then we have

\[
\mathbf{H} = -\nabla \phi,
\]

where \( \phi \) is a scalar magnetic potential. Notice that \( \phi \) must be multivalued if the dielectric domain is non-simply connected.

The finite element methods suggested in the literature to solve these formulations are based on edge elements to approximate the electric and magnetic fields and nodal finite elements to discretize the magnetic scalar potential.

Several authors have studied the \( \mathbf{E} \) and \( \mathbf{H} - \phi \) formulations of the eddy current problem in three-dimensional bounded domains. In engineering journals we detach the papers of Albanese and Rubinacci [1], Bossavit and Verité [42], Golias et al. [62], Webb et al. [125, 126, 127], and Rodger and Eastham [118].

Analyses of these formulations from a mathematical point of view have been extensively done in the last decade. Alonso et al. [4, 5, 6, 7, 8] and Bermúdez et al. [19, 20, 21, 22, 23, 24] have dealt with magnetic and electric field formulations. The treatment of the eddy current problem in domains with general topology, the existence and uniqueness of weak formulations, the appropriate definition of boundary conditions and the convergence properties of the discrete problems are the main issues in these references.

We remark that industrial problems are closely related with some of the previous points, namely, the topology of the domain or the definition of appropriate boundary conditions. More precisely, in a general eddy current problem, we can distinguish several kind of conducting materials which can introduce different difficulties:

- **Thin wire coils**: wires with a small diameter, where the current density can be assumed constant on the whole section of the conductor. The current direction is also known and corresponds to the direction of the wire. This is for instance the case of a copper coil with several turns which is present in devices like induction heating furnaces.

- **Passive conductors**: solid conductors which are electrically supplied only by induction, i.e., submitted to a time variable field generated by coils or by other solid conductors.

- **Active solid conductors**: solid conductors through which electric current flows, possibly under differences of electric potential. In this case, it is necessary to prescribe either the voltages or the total current intensity through the conductor, but the current density distribution is not known and will be an unknown of the model.

The presence of the different sources must be taken into account by means of either boundary conditions or the right-hand side term in equation (5). For instance, many papers found in the literature assume that the domain only contains a coil and passive conductors. In that case, the only source is the known current density in the coil and there is no other source in conducting regions. This problem allows considering the domain \( \Omega_c \) totally included in \( \Omega \), which leads to an important simplification in the topology of the domain and specially, in boundary conditions.

However, in many industrial applications, an electric voltage or a current intensity is directly applied to the magnetic device. As we will see below, metallurgical electrodes are included in this group. In this case, the topology of the domain is also more complex because the conductors will not be totally included in the domain \( \Omega \).

Prescribing the voltage or current intensity on the boundary of the conducting domain in potential formulations have been handled in some publications. In particular, the formulation \( \mathbf{A} \mathbf{V} - \mathbf{A} \) allows imposing voltage by means of Dirichlet boundary conditions on the electric scalar potential and appropriate conditions on the magnetic vector potential (see for instance [26, 27]). However, if the current intensity is given, the \( \mathbf{T} - \phi \) formulation is
more convenient; for instance in [25, 27, 32], the authors propose to introduce the current by using an additional current vector potential, which can be obtained as the solution of a boundary value problem (see [27] for further details). Recently, Biro et al. [31] have proposed a technique to impose the voltage in the $T - \phi$ formulation.

Concerning the magnetic and electric field formulations, one of the main difficulties to study the problem in a general bounded domain is to define mathematically suitable and physically realistic boundary conditions. By working with $E$ or $H$, the natural and essential boundary conditions consist of giving the tangential component of the electric or the magnetic field ($E \times n$ or $H \times n$) on the boundary of the domain. In fact, these are the boundary conditions used for instance by Alonso et al. in [4, 5, 6, 7, 8] or Bermúdez et al. in [19].

However, essential and natural boundary conditions are not directly related with the physical data and do not allow imposing the input current or the voltage on the boundary of the conductor domain. In fact, the need of dealing only with the physical data in the simulation of industrial electrodes led us to study the eddy current problem by using more realistic boundary conditions. We remark that Bossavit has given in [40] a description of more general boundary conditions. The main objective of that paper concerns topological aspects and the introduction of homology tools to define these conditions, but not the mathematical study of the corresponding weak formulations and their discretization. This is why we have advanced in this area, trying to introduce directly the current intensity or the voltage on the boundary of the conducting materials, either in terms of the magnetic or the electric field and analyzing the resulting formulations [20, 21, 22, 23, 24].

The next section is devoted to introduce and analyze the eddy current problem by using realistic boundary conditions in terms of the magnetic or the electric fields.

3 NUMERICAL SOLUTION OF THE EDDY CURRENT PROBLEM SUBJECT TO REALISTIC BOUNDARY CONDITIONS

In this section we introduce different mathematical models to solve the eddy current problem with boundary conditions suitable from the point of view of applications. Although we often consider geometrical domains which arise in the simulation of electric arc furnaces, the analysis is valid for more general configurations.

First, we consider the eddy current problem on a domain containing conductors and dielectrics in which the only boundary data are the input current intensities entering the conductors. We introduce a weak formulation in terms of the magnetic field and impose the curl-free condition in the dielectric domain by means of a multivalued magnetic potential. We propose a finite element method to solve the problem, study its convergence properties, and discuss implementation issues.

Second, we consider a similar problem on a domain containing only conducting material. We introduce two weak formulations: one in terms of the magnetic field and the other in terms of the electric field. In each case, we propose finite element methods and study their convergence. We also consider the possibility of having potential differences as boundary data instead of input current intensities.

We refer the reader to [22, 23, 24, 119] to see detailed proofs of the results reported throughout this section.

We will use the same notation, $\Omega$, for the domains of all problems, although in general they are different. We do this to simplify the notation. However, no confusion should arise because only one domain is considered in each subsection.

We recall the definition of some function spaces which will be used throughout this
section. Let \( L^2(\Omega) \) be the space of square integrable complex valued functions defined in \( \Omega \):

\[
L^2(\Omega) := \left\{ f : \Omega \to \mathbb{C} : \int_{\Omega} |f|^2 < \infty \right\}.
\]

We denote by \( H^1(\Omega) \) the Sobolev space of functions belonging to \( L^2(\Omega) \) together with all their first order derivatives:

\[
H^1(\Omega) := \left\{ f \in L^2(\Omega) : \text{grad} f \in L^2(\Omega)^3 \right\}.
\]

The space of restrictions to \( \partial \Omega \) of such functions is denoted by \( H^{1/2}(\partial \Omega) \):

\[
H^{1/2}(\partial \Omega) := \left\{ W|_{\partial \Omega} : \text{with } W \in H^1(\Omega) \right\}.
\]

We also define the space of square integrable vector fields with a square integrable curl:

\[
H(\text{curl}, \Omega) := \left\{ G \in L^2(\Omega)^3 : \text{curl} G \in L^2(\Omega)^3 \right\},
\]

the natural norm of this space being

\[
\|G\|_{H(\text{curl},\Omega)} := \left[ \|G\|^2_{L^2(\Omega)^3} + \|\text{curl} G\|^2_{L^2(\Omega)^3} \right]^{1/2}.
\]

We recall that vector fields in \( H(\text{curl},\Omega) \) satisfies tangential continuity but not necessarily normal continuity. In fact, for any inner surface \( S \) in \( \Omega \) with unit normal vector \( n_S \), given \( G \in H(\text{curl},\Omega) \), the values of the tangential component of \( G, n_S \times (G \times n_S) \), must coincide at both sides of \( S \). Instead, the normal component of \( G, G \cdot n_S \), may have a jump through the surface \( S \).

Finally, we recall a notion that will be often used below. Given a sufficiently smooth scalar function \( W \) defined on a smooth surface \( S \) in the surface gradient of \( W \) on \( S \), which we denote \( \text{grad}_{S} W \), is the two-dimensional gradient of \( W \) in local orthonormal coordinates of \( S \). For a precise definition, we refer the reader for instance to [44]. However, all what we need in the sequel is the following characterization: Assume for simplicity that \( S \subset \partial \Omega \) and let \( W_* \) be any smooth extension of \( W \) to the whole \( \Omega \) (i.e., a smooth function \( W_* \) defined in \( \Omega \) and such that \( W_*|_{\partial \Omega} = W \)). The surface gradient of \( W \) coincides with the tangential component on \( S \) of the three-dimensional gradient of \( W_* \):

\[
\text{grad}_S W = n_S \times (\text{grad} W_* \times n_S) \quad \text{on } S.
\]

The right hand side above does not depend on the particular extension \( W_* \), but only on the values of \( W \) on \( S \).

3.1 A Finite Element Method in a Bounded Domain Containing Conductors and Dielectrics.

3.1.1 Statement of the problem

As we advanced above, to simulate the behavior of any electromagnetic device, we need to consider a domain including not only the device but also the media surrounding it. For instance, to simulate an electric furnace (see Section 4 below) we should include in the computational domain the electrodes, part of the wires supplying electric current, and the air around. This is why we start proposing a model to solve the eddy current problem in
a bounded domain $\Omega$ which includes conductors and dielectrics. We recall the equations in terms of electric and magnetic fields and current density:

\begin{align}
\text{curl } H &= J \quad \text{in } \Omega, \tag{11} \\
\omega H + \text{curl } E &= 0 \quad \text{in } \Omega, \tag{12} \\
\text{div}(\mu H) &= 0 \quad \text{in } \Omega, \tag{13} \\
\text{div}(\epsilon E) &= q \quad \text{in } \Omega, \tag{14} \\
J &= \sigma E \quad \text{in } \Omega. \tag{15}
\end{align}

We consider a simply-connected three-dimensional bounded domain $\Omega$ which consists of two parts, $\Omega_\text{C}$ and $\Omega_\text{D}$, occupied by conductors and dielectrics, respectively. For the sake of clarity we refer to the configuration shown in Figure 1, which corresponds to a sketch of a metallurgical electric arc furnace. We denote $\Omega_{\text{C}}^1, \ldots, \Omega_{\text{C}}^N$ the connected components of $\Omega_\text{C}$, which, in this example, correspond to the different electrodes and part of the wires carrying the electric current to the furnace.

![Sketch of an electric arc furnace.](image)

The domain $\Omega$ is assumed to have a Lipschitz-continuous simply-connected boundary $\partial \Omega$, which splits into two parts: $\partial \Omega = \Gamma_\text{C} \cup \Gamma_\text{D}$, with $\Gamma_\text{C} := \partial \Omega_\text{C} \cap \partial \Omega$ and $\Gamma_\text{D} := \partial \Omega_\text{D} \cap \partial \Omega$ being the outer boundaries of the conducting and dielectric domains, respectively. We denote $\Gamma_1 := \partial \Omega_\text{C} \cap \partial \Omega_\text{D}$, the interface between dielectrics and conductors. We also denote by $\mathbf{n}$ the outer unit normal vector to $\partial \Omega$.

We assume that the outer boundary of each electrode, $\partial \Omega_{\text{C}}^n \cap \partial \Omega$ ($n = 1, \ldots, N$), has two connected components, both with non-zero measure: the current entrance, $\Gamma_{\text{C}}^n$, where the electrode is connected to a wire supplying alternating electric current, and the electrode tip, $\Gamma_{\text{D}}^n$, where the electric arc arises. Finally, we denote $\Gamma_{\text{C}} := \Gamma_{\text{C}}^1 \cup \cdots \cup \Gamma_{\text{C}}^N$ and $\Gamma_{\text{D}} := \Gamma_{\text{D}}^1 \cup \cdots \cup \Gamma_{\text{D}}^N$, and notice that $\Gamma_{\text{C}} \cap \Gamma_{\text{D}} = \emptyset$. 
Following Bossavit [40], we consider the following boundary conditions:

\[
\begin{align*}
E \times n = 0 & \quad \text{on } \Gamma_w, \\
\int_{\Gamma_j^n} \text{curl } H \cdot n = I_n, & \quad n = 1, \ldots, N, \\
E \times n = 0 & \quad \text{on } \Gamma_j, \\
\mu H \cdot n = 0 & \quad \text{on } \partial \Omega,
\end{align*}
\]

where the only data are the current intensities \(I_n\) through each wire.

Condition (16) is the natural one to model the current free exit on the electrode tips. Conditions (17) account for the input intensities through each wire. Conditions (18) and (19) have been proposed by Bossavit in [40] in a more general setting. They will appear as natural boundary conditions of the weak formulation of the problem. The former implies the assumption that the electric current is normal to the surface on the current entrance, whereas the latter means that the magnetic field is tangential to the boundary. Of course, conditions (18) and (19) are not always thoroughly fulfilled, but they provide good approximations to what actually happens in many practical cases.

To numerically solve the eddy current model with the boundary conditions (16)–(19), we start analyzing a weak formulation of the problem involving only the magnetic field. We advance that this formulation will require to impose explicitly the curl-free condition, \(\text{curl } H = 0\) in the space of trial and test functions in the dielectric domain. Thus, to deal with this condition at a discrete level, following Bossavit and Verité [42], we will introduce a multivalued scalar magnetic potential in the domain occupied by the dielectrics.

### 3.1.2 A magnetic field formulation

In this section we obtain a weak formulation in terms of the magnetic field of equations (11)–(15), subject to the boundary conditions (16)–(19), and report some results concerning this formulation.

To this aim, we start by testing equation (12) with sufficiently smooth vector fields \(G\):

\[
i \omega \int_{\Omega} \mu H \cdot \tilde{G} + \int_{\Omega} \text{curl } E \cdot \tilde{G} = 0.
\]

Since the conductivity \(\sigma\) vanishes in the dielectric domain, because of (11) and (15), the magnetic field \(H\) is curl-free in \(\Omega_d\). Hence, it is enough to consider in (20) test functions satisfying

\[
\text{curl } G = 0 \quad \text{in } \Omega_d.
\]

Moreover, by virtue of the boundary condition (17), we further restrict the test functions to those satisfying analogous homogeneous boundary conditions:

\[
\int_{\Gamma_j^n} \text{curl } G \cdot n = 0, \quad n = 1, \ldots, N.
\]

On the other hand, the boundary condition (19) implies that the tangential component of the electric field \(E\) on \(\partial \Omega\) is a surface gradient. Indeed, after integrating \(i \omega \mu H \cdot n\) on any surface \(S\) contained on \(\partial \Omega\), by using (12) and Stokes Theorem we obtain

\[
0 = i \omega \int_{S} \mu H \cdot n = - \int_{S} \text{curl } E \cdot n = - \int_{\partial S} E \cdot t = - \int_{\partial S} n \times (E \times n) \cdot t,
\]
where $\mathbf{t}$ is a unit vector tangent to the curve $\partial S$. Therefore, since $\partial \Omega$ is simply-connected, we can assert that there exists a surface potential of the tangential component of $\mathbf{E}$; namely, a sufficiently smooth scalar function $V$ defined on $\partial \Omega$ up to a constant and such that
\[
\mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = -\nabla V \quad \text{on } \partial \Omega.
\]
Moreover, (16) and (18) imply that $V$ must be constant on each connected component of $\Gamma_j$ and $\Gamma_k$. Furthermore, in the case of the electric furnace, we may assume that the potential is the same on the whole $\Gamma_k$. (For the sake of clarity, we focus on this case and refer the reader to Remark 3.1 below for more general situations.) Hence $V$ can be chosen to be null on $\Gamma_k$. Then we can transform the second term of (20) by using a Green’s formula as follows:
\[
\begin{align*}
\int_\Omega \text{curl} \mathbf{E} \cdot \mathbf{G} &= \int_\Omega \mathbf{E} \cdot \text{curl} \mathbf{G} - \int_{\partial \Omega} \mathbf{E} \times \mathbf{n} \cdot \mathbf{G} = \int_\Omega \mathbf{E} \cdot \text{curl} \mathbf{G},
\end{align*}
\]
the latter because, if $V_*$ is any smooth extension of $V$ to the whole $\Omega$, by using Green’s formulas again we have
\[
\begin{align*}
\int_{\partial \Omega} \mathbf{E} \times \mathbf{n} \cdot \mathbf{G} &= -\int_{\partial \Omega} \nabla V_\ast \times \mathbf{n} \cdot \mathbf{G} = \int_\Omega \nabla V_\ast \cdot \text{curl} \mathbf{G} = \int_{\partial \Omega} V \text{curl} \mathbf{G} \cdot \mathbf{n} = 0,
\end{align*}
\]
where, in the last equality, we have used that $V_\ast = 0$ on $\Gamma_k$, $V$ is constant on each $\Gamma_j$, (21), and (22).

Now, by substituting (23) in (20) we obtain
\[
\begin{align*}
i\omega \int_\Omega \mu \mathbf{H} \cdot \mathbf{G} + \int_\Omega \mathbf{E} \cdot \text{curl} \mathbf{G} &= 0.
\end{align*}
\]
Moreover, because of (21) the second integral above reduces to the conducting domain $\Omega_c$, where (11) and (15) lead to $\mathbf{E} = \frac{1}{\sigma} \text{curl} \mathbf{H}$. Thus, we finally obtain
\[
\begin{align*}
i\omega \int_\Omega \mu \mathbf{H} \cdot \mathbf{G} + \int_{\Omega_c} \frac{1}{\sigma} \text{curl} \mathbf{H} \cdot \text{curl} \mathbf{G} &= 0.
\end{align*}
\]

Remark 3.1 We notice that, in general, the (constant) electric potentials on each connected component of $\Gamma_k$ cannot be assumed to be equal. In such a case they have to be prescribed as additional boundary conditions on each of these connected components, $\Gamma_1, \ldots, \Gamma_N$, except on one of them. This leads to a new term in the right-hand side of the weak formulation (25), namely,
\[
\begin{align*}
i\omega \int_\Omega \mu \mathbf{H} \cdot \mathbf{G} + \int_{\Omega_c} \frac{1}{\sigma} \text{curl} \mathbf{H} \cdot \text{curl} \mathbf{G} &= \sum_{n=1}^{N-1} \int_{\Gamma_k} V_n \text{curl} \mathbf{G} \cdot \mathbf{n},
\end{align*}
\]
where $V_n$, $n = 1, \ldots, N - 1$, are the corresponding prescribed constant potential differences with respect to the potential on $\Gamma_k^N$, which is taken as zero.

Next, we summarize the main theoretical results proved in [19, 23] for this problem. We introduce the following subspace of $H(\text{curl}, \Omega)$:
\[
\mathcal{X} := \left\{ \mathbf{G} \in H(\text{curl}, \Omega) : \text{curl} \mathbf{G} = 0 \text{ in } \Omega_c \right\}.
\]
Then the weak formulation (25) can be written as follows:

**Problem MP.** Given $I_1, \ldots, I_N \in \mathbb{C}$, find $H \in \mathcal{X}$ satisfying

$$
\int_{\Gamma_j} \mathbf{curl} H \cdot n = I_n, \quad n = 1, \ldots, N,
$$

and

$$
i\omega \int_{\Omega} \mu \mathbf{H} \cdot \mathbf{G} + \int_{\Omega_c} \frac{1}{\sigma} \mathbf{curl} H \cdot \mathbf{curl} \mathbf{G} = 0
$$

for all test functions $G \in \mathcal{X}$ such that $\int_{\Gamma_j} \mathbf{curl} G \cdot n = 0$, $n = 1, \ldots, N$.

Problem MP has been proved in [23] to be well posed in the following sense:

**Theorem 3.1** Given $I_1, \ldots, I_N \in \mathbb{C}$, problem MP has a unique solution $H$.

Once the magnetic field $H$ is known, the current density $J$ and the electric field $E$ can be readily computed in the conductors by means of (11) and (15), respectively. These are the magnitudes actually needed in most applications.

On the other hand, the following theorem (proved in [23]) shows that the solution of problem MP actually satisfies Maxwell equations (11)–(13) and the boundary conditions (16)–(19).

**Theorem 3.2** Let $H \in \mathcal{X}$ be the solution of problem MP. Let $J = \mathbf{curl} H$ and $E = \left(\frac{1}{\sigma} J\right)|_{\Omega_C}$. The following properties hold true:

$$
\begin{align*}
\text{div}(\mu \mathbf{H}) &= 0 & \text{in } \Omega, \\
i\omega \mu \mathbf{H} + \mathbf{curl} E &= 0 & \text{in } \Omega_C, \\
J &= 0 & \text{in } \Omega_D, \\
\int_{\Gamma_j} \mathbf{curl} H \cdot n &= I_n, \quad n = 1, \ldots, N, \\
\mu \mathbf{H} \cdot n &= 0 & \text{on } \partial\Omega.
\end{align*}
$$

Moreover, there exists a scalar function $V$ defined on $\partial\Omega$ such that

$$(\mathbf{E} \times n)|_{\Gamma_j} = V_n \text{ (constant)}, \quad n = 1, \ldots, N,$
$$

and

$$(\mathbf{E} \times n)|_{\Gamma_n} = 0 \quad \text{on } \Gamma_j, \quad n = 1, \ldots, N.$$

**Remark 3.2** The physical meaning of $V_n$ is the difference of electric potential between $\Gamma_j^n$ and $\Gamma_j^e$.

Notice that the theorems above show that problem MP allows us to determine uniquely the electric field $E$ in conductors. Instead, this field is not uniquely determined in the dielectrics (see Remark 3.2 in [19]). This is not a drawback in most applications where the typical goal is to model the behavior of conductors.

We refer the reader to the recent paper by Alonso Rodriguez et al. [7], which gives important contributions to the existence and uniqueness of the electric and magnetic field for general multi-connected domains; in particular, the well-posedness of several formulations appearing in the literature are checked in that paper.
3.1.3 Introducing a magnetic potential

Next, we show how problem MP can be transformed by introducing a (scalar) magnetic potential in the dielectric domain $\Omega_D$. To this aim, we follow Bossavit and Verité [42]. On the other hand, we will use important results proved in detail by Amrouche et al. in [10], to take into account the multiply-connected character of conductor and dielectric regions.

We assume that for each connected component of the conducting domain, $\Omega_n^p$, there exists a connected “cut” surface $\Sigma_n \subset \Omega_n^p$ such that $\partial \Sigma_n \subset \partial \Omega_n^p$ and $\tilde{\Omega}_n := \Omega_n^p \setminus \bigcup_{n=1}^{N} \Sigma_n$ is pseudo-Lipschitz and simply-connected (see, for instance, [10]). We also assume that $\Sigma_n \cap \Sigma_m = \emptyset$ for $n \neq m$ (see Figure 1) and that the boundary of each entrance surface, $\Gamma_n$, is a simple closed curve, that we denote $\gamma_n$.

We denote the two faces of each $\Sigma_n$ by $\Sigma_n^-$ and $\Sigma_n^+$, and fix a unit normal $\mathbf{n}_n$ on $\Sigma_n$ as the ‘outer’ normal to $\tilde{\Omega}_n$ along $\Sigma_n^+$. We choose an orientation for each $\gamma_n$ by taking its initial and end points on $\Sigma_n^-$ and $\Sigma_n^+$, respectively. We denote by $\mathbf{t}_n$ the unit vector tangent to $\gamma_n$, according to this orientation.

For any function $\tilde{\Psi} \in H^1(\tilde{\Omega}_n)$, we denote by

$$[\tilde{\Psi}]_{\Sigma_n} := \tilde{\Psi}|_{\Sigma_n^-} - \tilde{\Psi}|_{\Sigma_n^+}$$

the jump of $\tilde{\Psi}$ through $\Sigma_n$ along $\mathbf{n}_n$. The gradient of $\tilde{\Psi}$ can be seen as a vector field in $L^2(\tilde{\Omega}_n)^3$ and will be denoted by $\nabla_{\text{ad}} \tilde{\Psi}$.

Let $\Theta$ be the linear subspace of $H^1(\tilde{\Omega}_n)$ defined by

$$\Theta = \left\{ \tilde{\Psi} \in H^1(\tilde{\Omega}_n) : [\tilde{\Psi}]_{\Sigma_n} = \text{constant}, \ n = 1, \ldots, N \right\}.$$  

Then, for $\Psi \in H^1(\tilde{\Omega}_n)$ we have that $\nabla_{\text{ad}} \tilde{\Psi} \in H(\text{curl}, \Omega_n)$ if and only if $\tilde{\Psi} \in \Theta$, in which case $\text{curl}(\nabla_{\text{ad}} \tilde{\Psi}) = 0$ (see Lemma 3.11 in [10]).

Therefore, given $G \in H(\text{curl}, \Omega)$, $G$ is curl-free in $\Omega_D$ if and only if it admits a multi-valued scalar potential $\Psi \in \Theta$:

$$G \in \mathcal{X} \iff \exists \tilde{\Psi} \in \Theta : G = \nabla_{\text{ad}} \tilde{\Psi} \quad \text{in } \Omega_D.$$  

Moreover, since vector fields in $H(\text{curl}, \Omega)$ have continuous tangential components, the following compatibility condition must hold true:

$$G|_{\Omega_C} \times \mathbf{n} = \nabla_{\text{ad}} \tilde{\Psi} \times \mathbf{n} \quad \text{on } \Gamma_v.$$  

When a multivalued magnetic potential is used in the dielectric domain, the boundary conditions (17) can be imposed by fixing its jumps on the cut surfaces. Indeed, if $G \in H(\text{curl}, \Omega_D)$, $\tilde{\Psi} \in \Theta$ and $G \times \mathbf{n} = \nabla_{\text{ad}} \tilde{\Psi} \times \mathbf{n}$ on $\Gamma_v$, we have

$$\int_{\Gamma_v^+} \text{curl} G \cdot \mathbf{n} = \int_{\gamma_n} G \cdot \mathbf{t}_n = \int_{\gamma_n} \nabla_{\text{ad}} \tilde{\Psi} \cdot \mathbf{t}_n = [\tilde{\Psi}]_{\Sigma_n},$$

where we have used Stokes Theorem and the fact that $G \times \mathbf{n} = \nabla_{\text{ad}} \tilde{\Psi} \times \mathbf{n}$ on $\gamma_n \subset \Gamma_v$.

Because of this, problem MP can be rewritten in terms of the magnetic potential as follows:

**Problem HP.** Given $I_1, \ldots, I_N \in \mathbb{C}$, find $\mathbf{H} \in H(\text{curl}, \Omega_D)$ and $\tilde{\Phi} \in \Theta$ satisfying

$$[\tilde{\Phi}]_{\Sigma_n} = I_n, \quad n = 1, \ldots, N,$$

$$\mathbf{H} \times \mathbf{n} = \nabla_{\text{ad}} \tilde{\Phi} \times \mathbf{n} \quad \text{on } \Gamma_v.$$
and
\[ i\omega \int_{\Omega_C} \mu \mathbf{H} \cdot \mathbf{\tilde{G}} + i\omega \int_{\Omega_D} \mu \mathbf{\tilde{g}} \mathbf{\nabla} \tilde{\Phi} \cdot \mathbf{\nabla} \tilde{\Psi} + \int_{\Omega_C} \frac{1}{\sigma} \mathbf{\nabla} \times \mathbf{H} \cdot \mathbf{\nabla} \times \mathbf{G} = 0 \]

for all test functions \( \mathbf{G} \in H(\mathbf{curl}, \Omega_C) \) and \( \tilde{\Psi} \in \Theta \) such that \( \mathbf{G} \times \mathbf{n} = \mathbf{\nabla} \times \tilde{\Psi} \times \mathbf{n} \) on \( \Gamma_i \) and \( [\tilde{\Psi}]_{\Sigma_n} = 0 \), \( n = 1, \ldots, N \).

Let us remark that the magnetic potential \( \tilde{\Phi} \) is uniquely defined up to an additive constant.

Problem \( \text{HP} \) is the hybrid formulation magnetic field/scalar magnetic potential introduced by Bossavit and Vérité in [42], adapted to the eddy current problem with input current intensities as boundary data. The main advantage with respect to problem \( \text{MP} \) lies in the fact that a vector field is replaced by a scalar one in the dielectric domain, which, after discretization, leads to an important saving in computational effort. However, the generation of cuts for complex geometries can be a hard task. Some papers are devoted to this point and propose algorithms for the generation of cuts (see for instance [64, 82, 115]).

This formulation has been considered in different publications. Bossavit and Vérité have introduced it in [42] to propose a BEM-FEM method for solving the eddy current problem. Bermúdez \textit{et al.} in [19] and Alonso Rodríguez \textit{et al.} in [6] have analyzed it for problems with essential and natural boundary conditions. The main difference between these two papers lies on the treatment of the scalar potential in non-simply connected domains. In [6], the potential is decomposed into a continuous part plus a singular one which is explicitly constructed, while in [19] the multivalued potential is directly computed as described in this section. Moreover, non-homogeneous essential boundary conditions on the boundary of the dielectric domain has been considered in [19] by means of Lagrange multipliers.

Other possibilities to avoid the troublesome generation of cuts can be found in the literature. In [116], the holes of the conducting parts are filled with fake conductors of very low conductivity; however, by using this technique the matrix system can be ill-conditioned and the convergence of the problem becomes an important drawback. Recently, Alonso Rodríguez \textit{et al.} have proposed in [8] another approach which consists of imposing the curl-free condition in the dielectric domain by means of Lagrange multipliers. The resulting mixed formulation has been analyzed for problems with essential boundary conditions, but not when the data are the input current intensities.

### 3.1.4 Numerical solution

In this section, we introduce a discretization of problem \( \text{MP} \) and prove its convergence. Then we show that the obtained discrete problem is equivalent to a more convenient discrete version of problem \( \text{HP} \). Finally we discuss some implementation issues.

We will employ “edge” finite elements to approximate the magnetic field, more precisely, the lowest-order finite elements of the family introduced by Nédélec in [102].

We assume that \( \Omega, \Omega_C, \) and \( \Omega_D \) are polyhedra and consider regular tetrahedral meshes \( T_h \) of \( \Omega \), such that each element \( K \in T_h \) is completely contained either in \( \Omega_C \) or in \( \Omega_D \) (\( h \) stands as usual for the corresponding mesh-size).

The magnetic field is approximated in each tetrahedron \( K \) by finite elements of the form
\[
G_h(x) = a_K \times x + b_K, \quad a_K, b_K \in \mathbb{C}^3, \quad x \in K.
\]

An explicit computation shows that vector fields of this type have constant tangential components along each straight line in the Euclidean space. Moreover, the tangential components along the edges of \( K \) can be taken as the degrees of freedom defining one such finite element.
These elements are $H(\text{curl})$-conforming in the sense that the tangential traces on each triangular face $T$ of $K$ depend only on the degrees of freedom of $G_h$ on the three edges of $T$. So, if we set

$$
\mathcal{N}_h(\Omega) := \left\{ G_h \in H(\text{curl}, \Omega) : \text{curl}G_h = 0 \text{ in } \Omega_D \right\},
$$

the elements of this space are piecewise linear vector fields with tangential traces continuous through the faces of the mesh. We notice that although tangential continuity is imposed, normal continuity is not. These are the lowest-order Nédélec finite elements introduced in [102]. They are called “edge” elements because the unknowns are related with the edges of the mesh. We refer to [61] for a detailed mathematical analysis and to [38] for useful implementation issues.

We introduce the following finite-dimensional subspace of $X$:

$$
X_h := \left\{ G_h \in \mathcal{N}_h(\Omega) : \text{curl}G_h = 0 \text{ in } \Omega_D \right\},
$$

and define the following discrete version of problem MP:

**Problem DMP.** Given $I_1, \ldots, I_N \in \mathbb{C}$, find $H_h \in X_h$ satisfying

$$
\int_{\Gamma^n} \text{curl} H_h \cdot \mathbf{n} = I_n, \quad n = 1, \ldots, N,
$$

and

$$
i\omega \int_{\Omega} \mu H_h \cdot \mathbf{G}_h + \int_{\Omega_D} \frac{1}{\sigma} \text{curl} H_h \cdot \text{curl} \mathbf{G}_h = 0
$$

for all test functions $G_h \in X_h$ such that $\int_{\Gamma^n} \text{curl} G_h \cdot \mathbf{n} = 0$, $n = 1, \ldots, N$.

The following theorem shows the existence and uniqueness of the solution for this discrete problem as well as an optimal order of convergence under a smoothness assumption on the problem MP.

**Theorem 3.3** Let $H$ be the solution of problem MP. If $H|_{\Omega_C} \in H^1(\Omega_C)^3$, $H|_{\Omega_D} \in H^1(\Omega_D)^3$, and $\text{curl} H|_{\Omega_C} \in H^1(\Omega_C)^3$, then problem DMP has a unique solution $H_h$ which satisfies

$$
\|H - H_h\|_{H(\text{curl}, \Omega)} \leq C h \left[ \|H\|_{H^1(\Omega_C)^3} + \|\text{curl} H\|_{H^1(\Omega_C)^3} + \|H\|_{H^1(\Omega_D)^3} \right],
$$

where $C$ is a strictly positive constant independent of $h$.

**Remark 3.3** We refer to Theorem 16 of [23] for a similar result under milder smoothness assumptions.

Problem DMP is just a ‘theoretical’ discrete problem. In fact, for its actual implementation it is necessary to impose somehow the curl-free condition in $\Omega_D$ to trial and test functions. In what follows we show how to do it efficiently by introducing a discrete multivalued magnetic potential in the dielectric domain.

To this aim, we introduce an approximation of the space $\Theta$. We assume that the cut surfaces $\Sigma_n$ are polyhedral and the meshes are compatible with them, in the sense that each $\Sigma_n$ is a union of faces of tetrahedra $K \in \mathcal{T}_h$. Therefore, $\mathcal{T}_h^{\Sigma} := \{K \in \mathcal{T}_h : K \subset \Omega_\Sigma\}$ can
also be seen as a mesh of $\tilde{\Omega}_D$. Let $S_h(\tilde{\Omega}_D)$ be the space of standard linear finite elements defined on this mesh:

$$S_h(\tilde{\Omega}_D) := \left\{ \tilde{\Psi}_h \in H^1(\tilde{\Omega}_D) : \tilde{\Psi}_h|_K \text{ is linear for all } K \in \mathcal{T}_h(\tilde{\Omega}_D) \right\}.$$  

The functions in $S_h(\tilde{\Omega}_D)$ are in general discontinuous on the cut surfaces $\Sigma_n$ (i.e., their values on $\Sigma_n^+$ and $\Sigma_n^-$ may differ). Consider the finite-dimensional subspace of $\Theta$ given by

$$\Theta_h := \left\{ \Theta_h \in S_h(\tilde{\Omega}_D) : [\tilde{\Psi}_h]_{\Sigma_n} = \text{constant, } n = 1, \ldots, N \right\}.$$  

The following lemma, proved in [19], shows that each curl-free vector field in $\mathcal{N}_h(\Omega_D)$ admits a multivalued potential in $\Theta_h$.

**Lemma 3.1.** Let $G_h \in L^2(\Omega_D)^3$. Then $G_h \in \mathcal{N}_h(\Omega_D)$ with $\text{curl} \ G_h = 0$ in $\Omega_D$ if and only if there exists $\tilde{\Psi}_h \in \Theta_h$ such that $G_h = \text{grad} \tilde{\Psi}_h$ in $\Omega_D$. Such $\tilde{\Psi}_h$ is unique up to an additive constant.

According to this, problem DMP can be equivalently written in terms of the discrete magnetic potential as follows:

**Problem DHP.** Given $I_1, \ldots, I_N \in \mathbb{C}$, find $H_h \in \mathcal{N}_h(\Omega_D)$ and $\tilde{\Phi}_h \in \Theta_h$ satisfying

$$\tilde{\Phi}_h|_{\Sigma_n} = I_n, \quad n = 1, \ldots, N,$$

$$H_h \times n = \text{grad} \tilde{\Phi}_h \times n \quad \text{on } \Gamma,$$

and

$$\int_{\Omega_D} \mu H_h \cdot \tilde{G}_h + \int_{\Omega_D} \mu \text{grad} \tilde{\Phi}_h \cdot \text{grad} \tilde{\Psi}_h + \int_{\Omega_D} \frac{1}{\sigma} \text{curl} H_h \cdot \text{curl} \tilde{G}_h = 0$$

for all test functions $G_h \in \mathcal{N}_h(\Omega_D)$ and $\tilde{\Psi}_h \in \Theta_h$ such that $G_h \times n = \text{grad} \tilde{\Psi}_h \times n$ on $\Gamma$ and $[\tilde{\Psi}_h]_{\Sigma_n} = 0, \ n = 1, \ldots, N$.

The implementation of problem DHP requires to impose the following constraints:

- $G_h \times n = \text{grad} \tilde{\Phi}_h \times n \quad \text{on } \Gamma$,

- $[\tilde{\Psi}_h]_{\Sigma_n} = \text{constant, } n = 1, \ldots, N$ (which arise in the definition of $\Theta_h$).

To do this, we employ the following procedures (see [19] and [119] for more details).

For the first constraint we use that

$$\int_{\ell} G_h \cdot t_\ell = \int_{\ell} \text{grad} \tilde{\Psi}_h \cdot t_\ell = \tilde{\Psi}_h(P_\ell^+) - \tilde{\Psi}_h(P_\ell^-) \quad \forall \ell \text{ edge of } \mathcal{T}_h : \ell \subset \Gamma,$$

where $P_\ell^-$ and $P_\ell^+$ are the initial and end points of $\ell$, respectively, and $t_\ell$ is the unit tangent vector pointing from $P_\ell^-$ to $P_\ell^+$. Then the degrees of freedom of $G_h$ associated with the edges $\ell \subset \Gamma$ can be eliminated by static condensation in terms of those of $\tilde{\Psi}_h$ corresponding to the vertices of the mesh on $\Gamma$.

Regarding the second constraint, for each cut surface $\Sigma_n$ we in principle distinguish the degrees of freedom of $\tilde{\Psi}_h$ on $\Sigma_n^+$ from those on $\Sigma_n^-$. Then the latter are eliminated by using

$$[\tilde{\Psi}_h]_{\Sigma_n^-} = [\tilde{\Psi}_h]_{\Sigma_n^+} + [\tilde{\Psi}_h]_{\Sigma_n},$$

with $[\tilde{\Psi}_h]_{\Sigma_n} = I_n$ for trial functions and $[\tilde{\Psi}_h]_{\Sigma_n} = 0$ for test functions.
Remark 3.4 Notice that the electric potentials $V_n$ on $\Gamma^n$ do not appear in the previous formulation. An alternative approach that allows computing them is proposed in Remark 6 of [23]. It consists of solving the following discrete mixed problem:

Given $I_1, \ldots, I_N \in \mathbb{C}$, find $H_h \in \mathcal{N}_h(\Omega_c)$, $\tilde{\Phi}_h \in \Theta_h$, and $(V_1, \ldots, V_N) \in \mathbb{C}^N$ satisfying

$$H_h \times n = \nabla \Phi_h \times n \quad \text{on} \quad \Gamma_p,$$

$$\sum_{n=1}^N \int_{\Omega_n} \text{curl} \, H_h \cdot n \, W_n = \sum_{n=1}^N I_n W_n \quad \forall (W_1, \ldots, W_N) \in \mathbb{C}^N,$$

and

$$i \omega \int_{\Omega_c} \mu H_h \cdot \bar{G}_h + i \omega \int_{\Omega_p} \mu \nabla \Phi_h \cdot \nabla \tilde{\Psi}_h + \frac{1}{\sigma} \int_{\Omega_c} \text{curl} \, H_h \cdot \text{curl} \bar{G}_h + \sum_{n=1}^N \int_{\Gamma^n_p} \text{curl} \bar{G}_h \cdot n \, V_n = 0$$

for all test functions $G_h \in \mathcal{N}_h(\Omega_c)$ and $\tilde{\Psi}_h \in \Theta_h$ such that $G_h \times n = \nabla \Phi_h \times n$ on $\Gamma_c$.

In this case the intensities are not imposed by means of the jumps of the multivalued magnetic potential. Indeed, the constants $[\nabla \Phi_h]_{\Sigma_n}$ are additional unknowns that must be computed too, whereas the test functions $\tilde{\Psi}_h$ are taken in $\Theta_h$ without imposing $[\nabla \Phi_h]_{\Sigma_n} = 0$.

3.2 A Finite Element Method in a Bounded Domain Containing Only Conducting Materials.

The model described in the previous section is highly complex, specially because of the generation of meshes and cuts in three-dimensional domains. Therefore, it is interesting to have simpler models to describe separate components of the whole system. For instance, it is very useful to be able to approximate the electromagnetic fields only in conducting regions by using appropriate boundary conditions. Indeed, in the case of the electric furnace (see Figure 1), the eddy current problem can be solved in only one electrode, neglecting the influence of the other conducting parts. This allows important savings in computer time compared with the model of the whole furnace and still allows considering important three-dimensional effects.

In what follows we will propose a model to solve the eddy current problem in a particular bounded conducting domain as, for example, an electrode. We will analyze weak formulations of this problem in terms of either the magnetic field or the electric field, considering realistic boundary conditions from the point of view of applications in both cases. In particular, we will consider boundary conditions similar to those defined in the previous section, which are directly related with the input current intensities or the electric voltage. We will impose these boundary conditions by means of Lagrange multipliers and study the resulting mixed formulations as well as numerical methods to solve them.

3.2.1 Statement of the problem

Consider a bounded simply-connected conducting domain $\Omega$ with a Lipschitz-continuous simply-connected boundary $\partial \Omega$. This boundary splits into two surfaces: $\partial \Omega = \Gamma_0 \cup \Gamma_p$. The surface $\Gamma_p$ corresponds to the tip of the electrode where the electric arc arises; we assume $\Gamma_0$ is connected. The rest of the electrode boundary splits as follows:

$$\Gamma_j = \Gamma_j^0 \cup \Gamma_j^1 \cup \cdots \cup \Gamma_j^N,$$
where $\Gamma_j^n$, $n = 1, \ldots, N$, are those parts of the boundary connected to the wires supplying electric current to the electrode, and $\Gamma_j^0$ is the remaining part which we assume free of current flow (see Figure 2). We also assume $\Gamma_j^n \cap \Gamma_k^n = \emptyset$ and $\Gamma_j^n \cap \Gamma_m^m = \emptyset$, $m, n = 1, \ldots, N$, $m \neq n$. Finally, we denote by $\mathbf{n}$ the outer unit normal vector to $\partial\Omega$.

To solve the eddy current problem on the conducting domain $\Omega$, we consider again equations (11)–(15):

\begin{align*}
\text{curl } \mathbf{H} &= \mathbf{J} \quad \text{in } \Omega, \quad (26) \\
\mathbf{i} \omega \mu \mathbf{H} + \text{curl } \mathbf{E} &= \mathbf{0} \quad \text{in } \Omega, \quad (27) \\
\text{div}(\mu \mathbf{H}) &= 0 \quad \text{in } \Omega, \quad (28) \\
\text{div}(\varepsilon \mathbf{E}) &= q \quad \text{in } \Omega, \quad (29) \\
\mathbf{J} &= \sigma \mathbf{E} \quad \text{in } \Omega, \quad (30)
\end{align*}

with the conductivity $\sigma$ being now strictly positive in the whole domain $\Omega$, subject to the following boundary conditions:

\begin{align*}
\mathbf{E} \times \mathbf{n} &= 0 \quad \text{on } \Gamma_k^n, \quad (31) \\
\int_{\Gamma_j^n} \text{curl } \mathbf{H} \cdot \mathbf{n} &= I_n, \quad n = 1, \ldots, N, \quad (32) \\
\mathbf{E} \times \mathbf{n} &= 0 \quad \text{on } \Gamma_j^0, \quad (33) \\
\text{curl } \mathbf{H} \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \quad (34) \\
\mu \mathbf{H} \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \quad (35)
\end{align*}

where the only data are again the current intensities through each wire: $I_n$, $n = 1, \ldots, N$. Notice that compared with the boundary conditions used in the whole furnace, we have only added (34) which takes into account the fact that there is no current flow through $\Gamma_j^0$. 

![Figure 2. Sketch of an electrode.](image)
3.2.2 Analysis and numerical solution of a weak formulation in terms of the magnetic field

To obtain a weak formulation in terms of the magnetic field for the eddy current problem (26)–(30) with boundary conditions (31)–(35), we proceed in a similar way as we did in the previous section. In particular, we notice again that the boundary condition (35) together with equation (27) imply that the tangential component of $\mathbf{E}$ on the boundary of $\Omega$ is a surface gradient, namely, $\mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = -\text{grad}_\mathbf{E} \mathbf{V}$ on $\partial \Omega$, for some scalar function $V$ vanishing on $\Gamma_k$ because of (31). Furthermore, because of (33), $V$ must be constant on each $\Gamma_i^n$, $n = 1, \ldots, N$.

By virtue of (34) and (32) it is enough to test equation (27) with vector fields $\mathbf{G}$ satisfying $\text{curl} \mathbf{G} \cdot \mathbf{n} = 0$ on $\Gamma_j^0$ and $\int_{\Gamma_j^n} \text{curl} \mathbf{G} \cdot \mathbf{n} = 0$, $n = 1, \ldots, N$. For such $\mathbf{G}$, similar arguments to those used in the previous section to obtain (24) lead to $\int_{\partial \Omega} \mathbf{E} \times \mathbf{n} \cdot \mathbf{G} = 0$. Therefore, by using a Green’s formula and taking into account that $\mathbf{E} = \frac{i\omega}{\sigma} \text{curl} \mathbf{H}$, we obtain

$$i\omega \int_{\Omega} \mu \mathbf{H} \cdot \mathbf{G} + \int_{\Omega} \frac{1}{\sigma} \text{curl} \mathbf{H} \cdot \text{curl} \mathbf{G} = 0.$$  

Thus, we are led to the following variational problem:

**Problem PI.** Given $I_1, \ldots, I_N \in \mathbb{C}$, find $\mathbf{H} \in \mathcal{H}(\text{curl}, \Omega)$ satisfying

$$\int_{\Gamma_j^n} \text{curl} \mathbf{H} \cdot \mathbf{n} = I_n, \quad n = 1, \ldots, N,$$

$$\text{curl} \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_j^0,$$

and

$$i\omega \int_{\Omega} \mu \mathbf{H} \cdot \mathbf{G} + \int_{\Omega} \frac{1}{\sigma} \text{curl} \mathbf{H} \cdot \text{curl} \mathbf{G} = 0$$

for all test functions $\mathbf{G} \in \mathcal{H}(\text{curl}, \Omega)$ such that $\int_{\Gamma_j^n} \text{curl} \mathbf{G} \cdot \mathbf{n} = 0$, $n = 1, \ldots, N$, and $\text{curl} \mathbf{G} \cdot \mathbf{n} = 0$ on $\Gamma_j^0$.

Next, we summarize the main theoretical results proved in [24] for this model. The first one shows that problem PI is well posed:

**Theorem 3.4** Given $I_1, \ldots, I_N \in \mathbb{C}$, problem PI has a unique solution $\mathbf{H}$.

To impose the constraints in problem PI, we consider a mixed formulation. It consists of handling these boundary conditions in a weak sense by introducing a Lagrange multiplier: a scalar functions $V$ defined on $\partial \Omega$ and satisfying

- $V$ vanishes on $\Gamma_k$ and
- $V$ is constant on each $\Gamma_j^n$, $n = 1, \ldots, N$.

This scalar function must also satisfy a mild regularity assumption: it must be the restriction to $\Gamma_j$ of a function in $H^1(\Omega)$ (see [23] for details). Thus, the Lagrange multiplier is chosen from the space

$$\mathcal{L} := \left\{ W \in H^{1/2}(\partial \Omega) : W = 0 \text{ on } \Gamma_k \text{ and } W|_{\Gamma_j^n} = \text{constant, } n = 1, \ldots, N \right\}.$$
The resulting mixed problem associated with problem \textbf{PI} is the following:

**Problem MPI.** Given $I_1, \ldots, I_N \in \mathbb{C}$, find $H \in H(\text{curl}, \Omega)$ and $V \in \mathcal{L}$ satisfying

$$i\omega \int_{\Omega} \mu H \cdot \tilde{G} + \int_{\Omega} \frac{1}{\sigma} \text{curl} H \cdot \text{curl} \tilde{G} + \int_{\Gamma_j} \text{curl} \tilde{G} \cdot n \, V = 0 \quad \forall \tilde{G} \in H(\text{curl}, \Omega)$$

and

$$\int_{\Gamma_j} \text{curl} H \cdot n \, \bar{W} = \sum_{n=1}^{N} I_n \int_{\Gamma_j} \bar{W} \quad \forall W \in \mathcal{L}.$$  

The following theorem shows that problems \textbf{PI} and \textbf{MPI} are actually equivalent. Its proof is based on the classical Babuška-Brezzi theory. In particular, the inf-sup condition is proved by using results concerning vector potentials in $\mathbb{R}^3$ (see [24] for further details).

**Theorem 3.5** Given $I_1, \ldots, I_N \in \mathbb{C}$, let $H \in H(\text{curl}, \Omega)$ be the solution of problem \textbf{PI}. Then there exists a unique $V \in \mathcal{L}$ such that $(H, V)$ is the only solution of problem \textbf{MPI}.

Next theorem, which has been proved in [24] too, shows that the solution of problem \textbf{MPI} actually satisfies Maxwell equations (26)–(28) and the boundary conditions (31)–(35).

**Theorem 3.6** Given $I_1, \ldots, I_N \in \mathbb{C}$, let $(H, V)$ be the solution of problem \textbf{MPI}. Let $J = \text{curl} H$ and $E = \frac{1}{\sigma} J$. Then

$$i\omega \mu H + \text{curl} E = 0 \quad \text{in} \ \Omega,$$

$$\text{div}(\mu H) = 0 \quad \text{in} \ \Omega,$$

$$\int_{\Gamma_j} \text{curl} H \cdot n = I_n, \quad n = 1, \ldots, N,$$

$$\text{curl} H \cdot n = 0 \quad \text{on} \ \Gamma_j^0,$$

$$\mu H \cdot n = 0 \quad \text{on} \ \partial \Omega.$$

Moreover,

$$n \times (E \times n) = - \text{grad}_E V \quad \text{on} \ \partial \Omega,$$

and, consequently,

$$E \times n = 0 \quad \text{on} \ \Gamma_E \quad \text{and} \quad E \times n = 0 \quad \text{on} \ \Gamma_j^n, \quad n = 1, \ldots, N.$$

As a consequence of this theorem, we can assert that $V$ is a surface potential of the electric field $E$ on $\partial \Omega$. Notice that this potential is chosen so as to be null on $\Gamma_E$; moreover, it is constant on each current entrance $\Gamma_j^n, n = 1, \ldots, N$.

Next, we introduce a finite element method to solve the mixed problem \textbf{MPI} and study its convergence properties. To this goal, we assume that $\Omega$ is a polyhedron and that $\Gamma_{E}$ and $\Gamma_{\text{in}}, n = 0, \ldots, N$, are all polyhedral surfaces.

We consider shape-regular tetrahedral meshes $T_h$ of $\Omega$, compatible with the splitting of the boundary of the domain in the sense that $\forall K \in T_h$ with a face $T$ lying on $\partial \Omega$ either $T \subset \Gamma_E$ or $T \subset \Gamma_j^n$ for some $n = 0, \ldots, N$.

The magnetic field, which is a function of $H(\text{curl}, \Omega)$, is discretized again by the lowest-order Nédélec edge finite elements described in the previous section.
Let $T_h^{\partial \Omega}$ be the triangular mesh induced by $T_h$ on the polyhedral surface $\partial \Omega$. Then the Lagrange multiplier will be approximated in the following finite-dimensional subspace of $L$:

\[ L_h := n \mathcal{W}_h \mathcal{S}_h(\partial \Omega), \]

where $\mathcal{S}_h(\partial \Omega)$ is the space of piecewise linear continuous functions defined on the triangular mesh $T_h^{\partial \Omega}$.

Thus, we define the following discrete version of problem MPI:

**Problem DMPI.** Given $I_1, \ldots, I_N \in \mathbb{C}$, find $H_h \in \mathcal{N}_h(\Omega)$ and $V_h \in L_h$ satisfying

\[ \omega \int_{\Omega} \mu H_h \cdot \bar{G}_h + \int_{\Omega} \frac{1}{\sigma} \text{curl} \, H_h \cdot \text{curl} \, \bar{G}_h + \int_{\Gamma} \text{curl} \, \bar{G}_h \cdot n \, V_h = 0 \quad \forall G_h \in \mathcal{N}_h(\Omega) \]

and

\[ \int_{\Gamma} \text{curl} \, H_h \cdot n \, W_h = \sum_{n=1}^{N} \int_{\Gamma} I_n \, W_h \quad \forall W_h \in L_h. \]

Next theorem (proved also in [24]) shows that the resulting finite element approximation converges to the exact solution with optimal order.

**Theorem 3.7** Given $I_1, \ldots, I_N \in \mathbb{C}$, problem DMPI has a unique solution $(H_h, V_h)$. Furthermore, if the solution $(H, V)$ of problem MPI satisfies $H \in H^1(\Omega)^3$ and $\text{curl} \, H \in H^1(\Omega)^3$, then the following error estimate holds true:

\[ \|H - H_h\|_{H(\text{curl},\Omega)} \leq C h \left[ \|H\|_{H^1(\Omega)^3} + \|\text{curl} \, H\|_{H^1(\Omega)^3} \right]. \]

**Remark 3.5** We refer the reader to [24] to see more complete results concerning error estimates for the magnetic field and the Lagrange multiplier, and under milder assumptions.

### 3.2.3 Analysis and numerical solution of a weak formulation in terms of the electric field

The eddy current problem using electric current intensities as boundary data can be alternatively solved in terms of the electric field instead of the magnetic field. In this section we briefly describe the resulting formulation and refer the reader to [21, 22] for further details.

We denote $C := E \cup \bigcup_{n=1}^{N} \Gamma_n$, the union of the disconnected pieces of $\partial \Omega$ where $E \times n = 0$. According to (27) and (35), we notice that

\[ \text{curl} \, E \cdot n = -i\omega \mu H \cdot n = 0 \quad \text{on} \quad \partial \Omega. \]

Then, because of (31) and (33), $E$ belongs to the following closed subspace of $H(\text{curl}, \Omega)$:

\[ \mathcal{Y} := \left\{ F \in H(\text{curl}, \Omega) : \text{curl} \, F \cdot n = 0 \text{ on } \partial \Omega \text{ and } F \times n = 0 \text{ on } \Gamma_c \right\}. \]

Moreover, since from (26) and (30) $\text{curl} \, H = \sigma E$, we have

\[ \int_{\Omega} \sigma E \cdot \bar{F} = \int_{\Omega} \text{curl} \, H \cdot \bar{F} \quad \forall F \in \mathcal{Y}. \]
Consequently, from (27) we obtain

$$\int_{\Omega} \sigma \mathbf{E} \cdot \tilde{\mathbf{F}} + \int_{\Omega} \frac{1}{\omega \mu} \text{curl} \mathbf{E} \cdot \text{curl} \tilde{\mathbf{F}} = \int_{\Omega} \text{curl} \mathbf{H} \cdot \tilde{\mathbf{F}} - \int_{\Omega} \mathbf{H} \cdot \text{curl} \tilde{\mathbf{F}} \quad \forall \mathbf{F} \in \mathcal{Y}. \tag{36}$$

The right hand side above depends on the input intensities $I_1, \ldots, I_N \in \mathbb{C}$ because $\mathbf{H}$ does it. To make explicit this dependence, we introduce the magnetic fields $\mathbf{H}_n$, $n = 1, \ldots, N$, which are the solutions of problems $\text{PI}$ with input current intensities equal to 1 on $\Gamma_1^n$ and 0 on $\Gamma_m^n$, $m \neq n$. Since $\mathbf{H}$ depends linearly on $I_1, \ldots, I_N$, we have that $\mathbf{H} = \sum_{n=1}^{N} I_n \mathbf{H}_n$. Therefore, equation (36) can be equivalently written as follows:

$$\int_{\Omega} \sigma \mathbf{E} \cdot \tilde{\mathbf{F}} + \int_{\Omega} \frac{1}{\omega \mu} \text{curl} \mathbf{E} \cdot \text{curl} \tilde{\mathbf{F}} = \sum_{n=1}^{N} I_n L_n(\tilde{\mathbf{F}}) \quad \forall \mathbf{F} \in \mathcal{Y},$$

where $L_n$ is the linear functional on $H(\text{curl}, \Omega)$ defined by

$$L_n(\mathbf{F}) := \int_{\Omega} (\text{curl} \mathbf{H}_n \cdot \mathbf{F} - \mathbf{H}_n \cdot \text{curl} \mathbf{F}), \quad \mathbf{F} \in H(\text{curl}, \Omega).$$

Thus, we are led to the following variational problem which involves only the electric field:

**Problem PE.** Given $I_1, \ldots, I_N \in \mathbb{C}$, find $\mathbf{E} \in \mathcal{Y}$ satisfying

$$\int_{\Omega} \sigma \mathbf{E} \cdot \tilde{\mathbf{F}} + \int_{\Omega} \frac{1}{\omega \mu} \text{curl} \mathbf{E} \cdot \text{curl} \tilde{\mathbf{F}} = \sum_{n=1}^{N} I_n L_n(\tilde{\mathbf{F}}) \quad \forall \mathbf{F} \in \mathcal{Y}.$$

We advance that we will introduce an alternative more convenient way of evaluating the right hand side of this problem, which avoids computing the magnetic fields $\mathbf{H}_n$ present in the definition of $L_n(\tilde{\mathbf{F}})$.

The following result, which has been proved in [22], shows that this problem is well posed and its solution is the electric field we are looking for.

**Theorem 3.8** Given $I_1, \ldots, I_N \in \mathbb{C}$, problem PE has a unique solution $\mathbf{E} = \frac{1}{\sigma} \text{curl} \mathbf{H}$, with $\mathbf{H}$ being the solution of problem PI.

On the other hand, the following lemma (proved in [22] too) shows that, for each $\mathbf{F} \in \mathcal{Y}$, its tangential trace has a surface potential which takes constant values on each $\Gamma_j^n$ and, moreover, $L_n(\mathbf{F})$ is the difference of potential between $\Gamma_j^n$ and $\Gamma_k^n$.

**Lemma 3.2** For all $\mathbf{F} \in \mathcal{Y}$, there exists a unique scalar function $W$ defined on $\partial \Omega$, such that $\mathbf{n} \times (\mathbf{F} \times \mathbf{n}) = \text{grad}_\mathbf{n} W$ on $\partial \Omega$ and $W|_{\Gamma_k^n} = 0$. Moreover, $W|_{\Gamma_j^n}$ is constant and $L_n(\mathbf{F}) = W|_{\Gamma_j^n} - W|_{\Gamma_k^n}$, $n = 1, \ldots, N$.

According to this lemma, the magnetic fields $\mathbf{H}_n$ are not actually needed to evaluate the right hand side of problem PE for $\mathbf{F} \in \mathcal{Y}$, as far as the corresponding potentials $W|_{\Gamma_j^n}$ can be computed somehow. This fact will be used to define a finite element method to approximate its solution without the previous computation of $\mathbf{H}_n$. 
Since the electric field is a function of $H(\text{curl}, \Omega)$, it can be discretized again by Nédélec edge finite elements. Then, to approximate $E$, we will use the following finite-dimensional subspace of $\mathcal{Y}$:

$$\mathcal{Y}_h := \left\{ F_h \in \mathcal{N}_h(\Omega) : \text{curl} F_h \cdot n = 0 \text{ on } \partial \Omega \text{ and } F_h \times n = 0 \text{ on } \Gamma_c \right\}.$$ 

To avoid computing $L_n(F_h)$ for $F_h \in \mathcal{Y}_h$, we use Lemma 3.2 and compute instead the difference of potential associated to this field between $\Gamma_j^n$ and $\Gamma_j$. We do this by means of a line integral of the field $F_h$ on a curve joining this two surfaces. To this aim, we define the following linear functionals on $\mathcal{Y}_h$:

$$\tilde{L}_n(F_h) := \int_{c_n} F_h \cdot t_n, \quad F_h \in \mathcal{Y}_h,$$

with $c_n$ being a simple curve joining $\Gamma_j^n$ with $\Gamma_j$ and $t_n$ a unit tangent vector to $c_n$, $n = 1, \ldots, N$ (see Figure 3).

![Figure 3. Curves $c_n$ joining $\Gamma_j^n$ with $\Gamma_j$.](image)

The following lemma, proved in [22], shows that the linear operators $L_n$ and $\tilde{L}_n$ coincide on the discrete space $\mathcal{Y}_h$.

**Lemma 3.3** For all $F_h \in \mathcal{Y}_h$,

$$L_n(F_h) = \tilde{L}_n(F_h).$$

According to this lemma, we may conveniently consider the following discretization of problem $\textbf{PE}$, which does not involve the fields $H_n$, $n = 1, \ldots, N$, appearing in the definition of $L_n$.

**Problem DPE.** Given $I_1, \ldots, I_N \in \mathbb{C}$, find $E_h \in \mathcal{Y}_h$ satisfying

$$\int_{\Omega} \sigma E_h \cdot \tilde{F}_h + \int_{\Omega} \frac{1}{i \omega \mu} \text{curl} E_h \cdot \text{curl} \tilde{F}_h = \sum_{n=1}^{N} I_n \tilde{L}_n(F_h) \quad \forall F_h \in \mathcal{Y}_h.$$
Under convenient smoothness assumptions on the solution of the continuous problem PE, it has been proved in [22] that the finite element approximation obtained from this discrete problem converges with optimal order to the exact solution.

**Theorem 3.9** Given $I_1, \ldots, I_N \in \mathbb{C}$, let us assume that the solution $E$ of problem PE satisfies $E \in H^1(\Omega)^3$ and $\text{curl} E \in H^1(\Omega)^3$. Then problem DPE has a unique solution $E_h$ and the following error estimate holds true:

$$
\|E - E_h\|_{H(\text{curl}, \Omega)} \leq C h \left[ \|E\|_{H^1(\Omega)^3} + \|\text{curl} E\|_{H^1(\Omega)^3} \right].
$$

**Remark 3.6** We refer the reader to [22] for similar results under milder smoothness assumptions.

The implementation of problem DPE requires to impose the conditions $F_h \times n = 0$ on $\Gamma_C$ and $\text{curl} F_h \cdot n = 0$ on $\partial \Omega$, present in the definition of $\mathcal{Y}_h$. The first one can be easily imposed by taking $F_h$ with vanishing degrees of freedom on $\Gamma_C$. Since $F_h \times n = 0$ implies $\text{curl} F_h \cdot n = 0$, the first conditions is thus automatically imposed on $\Gamma_C$ too. However this is not the case on $\Gamma^0$. This has been done in [21, 22] by introducing a Lagrange multiplier, following techniques similar to those developed in the previous section to impose the condition $\text{curl} H \cdot n = 0$ on $\Gamma^0$. Convergence results for the resulting mixed problem have been proved in the same references.

**3.2.4 The eddy current problem with electric potential as boundary data. Analysis in terms of the electric field**

As we have remarked above, it is usual in industrial applications to know either the input current intensities or the electric potentials on the current entrances. In this section, we introduce the guidelines to study the eddy current problem (26)–(30) with electric potentials as boundary data.

We consider the boundary decomposition introduced in Section 3.2.1. Suppose that the boundary data are now the electric potentials on the surfaces $\Gamma^n_J, n = 1, \ldots, N,$ instead of the corresponding input current intensities. Thus, we have the following boundary conditions:

$$
\text{curl} H \cdot n = 0 \quad \text{on} \quad \Gamma^0_J, \quad (37)
$$
$$
\mu H \cdot n = 0 \quad \text{on} \quad \partial \Omega, \quad (38)
$$
$$
V = 0 \quad \text{on} \quad \Gamma^0_W, \quad (39)
$$
$$
V = V_n \quad \text{on} \quad \Gamma^n_J, \quad n = 1, \ldots, N, \quad (40)
$$

where the electric potentials $V_n$ on the surfaces $\Gamma^n_J, n = 1, \ldots, N$, are the only data of the problem. (Indeed, what must be known is the difference of potential between $\Gamma^n_J$ and $\Gamma^0_W$ because we have taken $V = 0$ on $\Gamma^0_W$.)

Our goal is to consider equations (26)–(30) subject to these boundary conditions, with the surface electric potential being related to the electric field by the standard relation $\mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = \text{grad} V$ on $\partial \Omega$. Notice that, as a consequence of (39) and (40), the electric field will satisfy also the boundary conditions

$$
E \times n = 0 \quad \text{on} \quad \Gamma^0_W, \quad (41)
$$
$$
E \times n = 0 \quad \text{on} \quad \Gamma^n_J, \quad n = 1, \ldots, N. \quad (42)
$$
To obtain a variational formulation, we notice that $E \in \mathcal{Y}$ as a consequence of (27), (38), (41), and (42). Hence, Lemma 3.2 applies to $E$, showing the existence of a surface electric potential $V$ and that $V|_{r=1} = L_n(E)$, $n = 1, \ldots, N$.

Therefore, the same arguments leading to problem PE show that the eddy current problem with potentials as boundary data can be written in terms of the electric field as follows:

**Problem PV.** Given $V_1, \ldots, V_N \in \mathbb{C}$, find $E \in \mathcal{Y}$ satisfying

$$L_n(E) = V_n, \quad n = 1, \ldots, N,$$

and

$$\int_\Omega \sigma E \cdot \tilde{F} + \int_\Omega \frac{1}{\omega \mu} \text{curl} E \cdot \text{curl} \tilde{F} = 0$$

for all test function $F \in \mathcal{Y}$ such that $L_n(F) = 0$, $n = 1, \ldots, N$.

This problem is well posed as it is shown in the following theorem, which has been proved in [22], too.

**Theorem 3.10** Given $V_1, \ldots, V_N \in \mathbb{C}$, problem PV has a unique solution.

To impose the constraint $L_n(E) = V_n$, we introduce the following mixed formulation:

**Problem MPV.** Given $V_1, \ldots, V_N \in \mathbb{C}$, find $E \in \mathcal{Y}$ and $(I_1, \ldots, I_N) \in \mathbb{C}^N$ satisfying

$$\int_\Omega \sigma E \cdot \tilde{F} + \int_\Omega \frac{1}{\omega \mu} \text{curl} E \cdot \text{curl} \tilde{F} - \sum_{n=1}^N I_n L_n(\tilde{F}) = 0 \quad \forall F \in \mathcal{Y}$$

and

$$- \sum_{n=1}^N L_n(E)K_n = - \sum_{n=1}^N V_n K_n \quad \forall(K_1, \ldots, K_N) \in \mathbb{C}^N.$$

The following theorem (proved also in [22]) shows that problems PV and MPV are actually equivalent.

**Theorem 3.11** Given potentials $V_1, \ldots, V_N \in \mathbb{C}$, let $E \in \mathcal{Y}$ be the solution of problem PV. Then there exists a unique $n$-tuple $(I_1, \ldots, I_N) \in \mathbb{C}^N$ such that $(E, (I_1, \ldots, I_N))$ is the only solution of problem MPV.

**Remark 3.7** Let $(E, (I_1, \ldots, I_N))$ be the unique solution of problem MPV. Then $E$ is the solution of problem PE with data $I_n$, $n = 1, \ldots, N$. Indeed, the first equation of problem MPV is exactly the equation of problem PE with these $I_n$, and PE has a unique solution.

Hence, $H = -\frac{1}{\omega \mu} \text{curl} E$ is also the unique solution of problem PI with data $I_n$, $n = 1, \ldots, N$. Thus, Theorem 3.6 shows that the solution $E$ of problem MPV satisfies Maxwell equations (26)–(28) and the boundary conditions (37) and (38). Moreover, the first equation of problem PV and Lemma 3.2 implies that $V_n$ are the values of a surface potential of $E$. Consequently the boundary conditions (41) and (42) also hold true.

**Remark 3.8** Nédélec edge finite elements can be used once more to discretize $E$ for the numerical solution of problems PV and MPV. After discretization, the linear functionals $L_n$ can be again substituted by $\tilde{L}_n$, which can be computed without the need of having computed $H_n$ in advance. We refer the reader to [21, 22] for more details on the numerical methods and their convergence properties.
4 NUMERICAL RESULTS. APPLICATION TO SIMULATION OF METALLURGICAL ELECTRODES

All of the numerical methods described in the previous sections have been implemented in MATLAB codes and have been applied to different problems with known analytical solutions in order to validate the computer codes and to test the convergence properties of the methods. Results of these numerical tests and experimental observations have been reported in [19, 22, 23, 24]).

The methods have also been applied to the numerical simulation of metallurgical electrodes under industrial working conditions. In this section we will show some numerical results corresponding to different kind of electrodes for silicon and ferro-silicon production. Before this, we start giving a description of the physical phenomena involved in an electric furnace and reviewing the literature concerning this topic.

Let us remark that the electromagnetic formulation described in Section 3.2.2 has been also coupled with the heat transfer equation to build a thermoelectrical model useful to simulate the behavior of metallurgical electrodes or other electrical devices. We refer the reader to [119] for further details and numerical results on this subject.

4.1 The Physical Problem

Silicon (Si) is the second most abundant element in the earth’s crust after oxygen. In natural form, it can be found mainly as silicon dioxide (Silica, SiO₂) and silicates (silicon combined with other elements). In particular, quartz and sand are two of the most common forms.

Silicon is produced industrially by reduction of silicon dioxide, as quartz or quartzite, with carbon, by a reaction which can be written in a simple way as follows ([120]):

\[ \text{SiO}_2 + 2\text{C} = \text{Si} + 2\text{CO} \]

This reaction takes place in reduction furnaces like those used for producing ferroalloys, calcium carbide, iron, etc. In particular, silicon is obtained in submerged “arc” furnaces which use three-phase alternating current. A simple sketch of the furnace can be seen in Figure 4. It consists of a cylindrical pot containing charge materials and three electrodes disposed conforming an equilateral triangle.

The electric current is supplied to the plant, usually at high voltage. Different transformers change the high voltage current into low-voltage high-intensity current suitable for the process. In modern furnaces the electric current is transformed separately for each phase. Figure 5 shows an example of an electric arrangement for a silicon furnace where three transformers with different phases are situated around the furnace. From the transformers, the electrical energy is conducted through bus bars to the electrodes which carry the electric current to the center of the furnace. At the tip of each electrode an electric arc is produced generating the high temperatures needed for the reduction chemical reactions to take place. This is why reduction furnaces are also called “arc furnaces”.

Electrodes are the main components of reduction furnaces. Typical diameter of electrodes is 1-2 m while their height is more than 10 m, their weight being greater than 10 mt.

Electric current enters each electrode through a metal ring which completely embraces the column, approximately 1 m above the charge level (see Figure 4). The ring is composed by different sections made of copper and called “contact clamps”. The electric current goes down crossing the column length comprised between the contact clamps and the lower end of the column generating heat by Joule effect. At the tip of the electrode an electric arc is produced, reaching temperatures of about 2500°C.

Classical electrodes extensively used in industry include pure graphite, prebaked and Söderberg electrodes. The latter are the most used in ferro-silicon industry. They are
composed by paste consisting of a carbon aggregate and a tar binding which are fed into a steel casing; the casing have steel fins attached to its inner part, which are placed radially in the cylinder (see Figure 7). The great amount of heat generated by Joule effect is partially employed to bake the paste; this is a crucial process during which the initially soft/liquid non-conductive paste at the top of the electrode becomes a solid conductor. The advantages of Söderberg electrodes with respect to pure graphite or prebaked electrodes are that they are built in larger sizes and cost less. However, as the electrode is consumed, it has to
be slipped and the steel casing moves with the carbon body, so it melts and pollutes silicon. This is why Söderberg electrodes cannot be used to obtain silicon metal or silicon with metallurgical quality, able to be used as alloying of other metals as aluminum. Thus prebaked electrodes have been, for many years, the only alternative for commercial silicon metal production.

In the early nineties, the Spanish company Ferroatlántica S.L. developed a new compound electrode named ELSA ([46]) which serves for the production of silicon metal. It seems to be the solution for all silicon furnaces because its cost can be up to one third the price of a prebaked electrode.

An ELSA electrode consists of a central column of baked carbonaceous material, graphite or similar, surrounded by a Söderberg-like paste (see Figure 6). There is a steel casing without fins that contains the paste until it is baked at the contact clamps zone. Two different slipping systems exist, one for the casing and another one for the central column; the combination of both systems is necessary so as to slip the casing as little as possible and also to carry out the correct extrusion of the carbon electrode. Then, unlike for Söderberg electrodes, the casing is not consumed and it is possible to produce silicon with metallurgical quality. The result is that the furnace operation is similar to that of prebaked electrodes, but the compound electrode is much less expensive. The disadvantage is that slipping velocity is not free as in prebaked electrodes, because the paste has to be baked before leaving the casing, so it is necessary a minimum period of time between slippages. Thus, the baking of paste is a crucial point in the working of this type of electrodes.

Figure 6. Sketch of an ELSA electrode.
4.2 Mathematical Modeling of Electrodes

In general, the design and parameters control of electrodes are very complex and numerical simulation plays an important role at this point. Modeling the involved phenomena in a computer allows analyzing the influence of changing a parameter without the need of expensive and difficult tests. Thus, an important number of mathematical models and computer programs have been developed during the last 20 years in order to understand and solve the main problems regarding electrode furnace operation.

These models have been devoted to compute different fields inside the electrode. In particular, temperature distribution, electromagnetic fields, and stress distribution have been obtained by solving the heat transfer equation, the Maxwell equations, and the elasticity equations, respectively. These equations are coupled because the source heat depends on electromagnetic fields, whereas physical parameters and stresses depend on temperature. This is why we can find in the literature several publications which deal with thermoelectrical and thermomechanical models.

The mathematical models based on cylindrical symmetry have been the most extensively used. These models neglect the effect of the two other electrodes (so-called “proximity effect”) and solve the mathematical problem in a vertical section of one electrode, by writing the equations in cylindrical coordinates. Another assumption of these models consists of considering axisymmetric boundary conditions which is not actually the case in the industrial situation as it will be detailed below.

During the last decades the axisymmetric models have been studied and applied for all types of electrodes. Indeed, concerning graphite pure electrodes, the most relevant mathematical models have been those focused on calculating the stress distribution in the joints between the graphite bars (see for instance [58]). D’Ambrosio et al. [54, 55, 56] have written a number of papers in which they study the temperature and the stress distribution in prebaked electrodes. They solve the axisymmetric models with a finite difference method. For Soderberg electrodes the Norwegian company ELKEM has developed several computer codes since 1970 to understand better the behavior of this type of electrodes. Let us mention an axisymmetric thermoelectrical model in transient conditions described in [72] and a thermomechanical one in [70]. The different models are shortly described in [74]. Conclusion and recommendations concerning Soderberg electrodes have been obtained by using these models and are discussed for instance in [73].

However, ELSA electrodes work in a different manner from classical ones. While classical electrodes have only a constitutive material, compound ones combine a good electric current conductor as graphite with a paste which becomes a good conductor only at high temperatures. The core of graphite is not only important in the movement of the column but also in the distribution of current inside the electrode. For instance, the skin effect typically present in all classic electrodes, only takes place at a comparable level at the tip of the compound ones. Other characteristic feature is that paste changes of state in some regions, as in the case of self-baked electrodes. On the other hand, unlike Soderberg electrodes, the non existence of fins leads to a geometrical axial symmetry.

In order to study the behavior of the ELSA electrodes, we have been working with Ferroatlántica I+D company from 1996 and developed several thermoelectrical and thermomechanical models. An axisymmetric finite element method to compute temperature and electric current distribution in a radial section of an ELSA electrode is described by Bermúdez et al. in [14, 15, 16, 17]. In particular, a steady model is introduced in [14], whereas theoretical mathematical results related to this problem can be found in [17]. A more complete transient model is studied in [16]. Finally, an axisymmetric thermomechanical model which allows us to compute the stress distribution is described in [15].

In despite of the simplifications, the two-dimensional models have given valuable infor-
mation on important electrode parameters and have been used for specific purposes with promising results. In particular, the position of the baked zone and the distribution of thermal stresses have been investigated at steady state and transient conditions. Factors like electric current intensity, slipping rate, shut-down/start-up procedures, material properties, etc, have been studied with axisymmetric models, too. The results obtained for ELSA electrodes have been presented in several metallurgical conferences during the last years. For instance, the effect of modifying the cooling water temperature, the size of the contact clamps or the characteristics of raw materials have been analyzed by Bullón et al. in [45].

The main advantage of these two-dimensional models compared with three-dimensional ones is of course saving of computing time. However, the assumption of cylindrical symmetry makes necessary to neglect the following facts:

- The electromagnetic effect caused on one electrode by the other two, the “proximity effect”. This effect arises because the magnetic field generated by each electrode induces eddy currents in the other two.

- Thermal boundary conditions are not axisymmetric. Indeed, the temperature of the air around the electrode is greater on the surfaces oriented toward the furnace center.

- Current entrance in the electrode through the contact clamps is not axisymmetric. Indeed, the current is transferred to the contact clamps through copper bus tubes which in its turn are connected to the transformers. In each electrode, half of the clamps receive current from one transformer while the other ones are connected to a second transformer (see Figure 5).

These points can be taken into account only by using a genuine three-dimensional model. Moreover, these models are needed to simulate Soderberg electrodes too, because they have steel fins to guide current into the central part of the electrode which break cylindrical symmetry. These fins are placed radially in the cylinder and welded to the casing, binding the baked carbon with the casing (see Figure 7 which shows cross sections of Soderberg and ELSA electrodes).

![Figure 7. Cross sections of ELSA and Soderberg electrodes.](image)

We notice that some of the above points have been studied as isolated problems. For instance, Böchmann and Olsen [36] have estimated the proximity effect for Soderberg electrodes based on an analytical approach developed in [57]. Another analytical method based on series expansions has been developed by Hot et al. [68] with the same goal. The proximity effect has also been numerically studied by using two-dimensional models solving
the electromagnetic problem on a horizontal section of the three electrodes. To this aim, a finite element method coupled with boundary elements has been used by Bermúdez et al. [13] to study the proximity effect in ELSA electrodes. For Söderberg ones, Pálsson and Jonsson [103] have proposed a finite element method in a bounded domain. However, these two-dimensional models are only valid in the lower part of the electrode, because they assume that the electric current is orthogonal to the considered two-dimensional section.

Thus, in order to take into account the different points neglected by the two-dimensional models, a three-dimensional model is actually needed. Only a few publications concerning three-dimensional simulation of metallurgical electrodes can be found in the literature. A finite difference approximation and a finite element method to study the thermo electrical behavior of Söderberg electrodes have been proposed by Innvær et al. [71] and by Ingason and Jonsson [77], respectively. Another finite difference method has been proposed by Zhou et al. in [128]. These papers are devoted to describe the problem from a modeling point of view. Geometry, equations, and boundary conditions are described and numerical results are reported. However, no mathematical analysis is included either of the continuous problem or of its numerical approximation. This is why we have focused our attention in the analysis and numerical solution of three-dimensional finite element methods, general enough to simulate any kind of electrodes and even the complete furnace.

The thermo electrical and thermomechanical simulation of electrodes involve the numerical solution of Maxwell equations. In particular, the alternating current and the low frequency (50 Hz) used in the furnace makes the eddy current model a good approximation to obtain the electromagnetic fields. In fact, as we have observed above, numerical solution of the eddy current problem in bounded domains represents an important difficulty which deserves special attention.

This is the reason why, although we have coupled the eddy current model with the heat transfer equation too (see for instance [119]), in next sections we focus our attention on results obtained with the numerical methods described in the previous section for the eddy current model.

4.3 Numerical Results for a Model of the Whole Furnace

We have applied the method described in Section 3.1 to simulate an electric furnace with three electrodes as shown in Figure 8. The geometrical data can be found in the same figure.

We have considered ELSA compound electrodes which consist of a cylindrical core of graphite and an outer part of Söderberg paste. See Figure 9 for geometrical data. For this model we have not considered the contact clamps and the casing. Moreover, we have assumed the Söderberg paste to be baked in the whole model domain. The electric current enters the electrodes through copper bars of rectangular section.

The physical parameters we have used are the following: \( \sigma = 10^6 \, (\Omega m)^{-1} \) for graphite; \( \sigma = 10^4 \, (\Omega m)^{-1} \) for Söderberg paste; \( \sigma = 0.5 \times 10^7 \, (\Omega m)^{-1} \) for copper; \( \mu = 4\pi \times 10^{-7} \, \text{Hm}^{-1} \); \( \omega = 2\pi \times 50 \, \text{Hz} \); one-phase intensities \( I_n = 7 \times 10^4 \, \text{A} \) for each electrode.

Figure 10 shows, on the left, the intensity of the computed current density \( |\mathbf{J}_h| := |\mathbf{curl} \, \mathbf{H}_h| \) on a horizontal section of the conductor domain (electrodes and wires); the same figure shows, on the right, the computed magnetic potential \( \Phi_h \) on a horizontal section of the dielectric domain (air). Figure 11 shows \( |\mathbf{J}_h| \) more in detail, in horizontal and vertical sections of one of the electrodes. Notice that the current distribution in the electrodes is not axisymmetric due to the proximity effect. These results are in good quantitative agreement with those obtained by using a simplified two-dimensional model described in [13], based on a BEM-FEM coupled method.
4.4 Numerical Results for a Model of One Electrode

In this section we show some results obtained with the numerical methods presented in Section 3.2 for ELSA and Söderberg electrodes. We have solved the electromagnetic problem in only one electrode, neglecting the proximity effect of the two others. We refer the reader to [119] for further details and results.

4.4.1 Numerical results for an ELSA electrode

To simulate an ELSA electrode, we have considered a computational cylindrical domain which contains the graphite, the paste, the casing, and the contact clamps which completely embrace the column. For the geometrical data see Figure 12. Since the aim of this simulation was to know the current distribution in the contact clamps zone, we have supposed that the electrode tip is flat. This is a suitable assumption, because the electrode is long enough for the influence of the tip in the contact clamps to be not significant. Figure 12 also shows, on the left, the used mesh. For further details see [119].

Although the proximity effect of the two other electrodes is neglected in this model, the problem is not axisymmetric because the electric current is introduced through 16 wires...
Moreover, as we noticed above, half of the clamps receive current from one transformer, while the other ones are connected to a second transformer with a different phase. For simplicity, we have supposed that the current enters the electrode by two tubes connected to the top of the contact clamps as shown in Figure 12. We have assumed input intensities of 35 kA entering through each port of current entrance at different phases.

Figures 13 and 14 show the modulus of the current density distribution on different cross sections of the electrode. The first one corresponds to horizontal sections at top and bottom of the contact clamps, while the second one shows a horizontal section 25 cm below the contact clamps and a vertical section. Notice that, in the current entrance area, the distribution is not axisymmetric. Instead, we observe almost cylindrical symmetry below the contact clamps. Therefore, the current distribution around the contact clamps does not seem to imply noticeable effects below them.

The interface between paste and graphite can be clearly seen in all these figures. Moreover, the obtained numerical results allow us to explain satisfactorily the electrical behavior
of the ELSA electrode. The electric current introduced in the electrode at the top of the contact clamps, circulates mainly on their outer surface and enters the electrode through the lower part of the clamps (see Figure 15). Accordingly, the greatest values of the Joule heating are reached near the bottom of the contact clamps as can be seen in the same figure.

Since the electrical resistivity of graphite is much lower than that of paste, once the current enters the electrode, it passes preferentially through the graphite core (see Figure 15). However, as the temperature of the electrode increases nearer its bottom, the electrical resistivity of paste diminishes and, about 0.5 m below the contact clamps, the amount of current going through the paste increases due to the skin effect (which anyway is quite small because the frequency is very low).

4.4.2 Numerical results for a Söderberg electrode

Finally, we show some results obtained for Söderberg electrodes. The thickness of the fins attached to the steel casing varies between 2 and 5 mm, so it is very difficult to include them in the mesh. To reduce the computational effort, we have assumed symmetry in the boundary conditions allowing us to simulate only part of the electrode, as shown in Figure 16. This figure shows the mesh corresponding to the simulated sector of the electrode, too. Further details can be found in [119].

Figure 17 shows the current density in two vertical sections, one between two fins and the other coinciding with one fin. Notice that the skin effect in Söderberg electrodes is more significant than in the ELSA electrode below clamps. Indeed, the highest values of the current density are attained on the outer surface of the electrode.

All of these results have been actually used to understand the behavior of the different kind of electrodes. We refer again the reader to [119] for further details, in particular concerning the thermoelectrical simulation.
Figure 12. Mesh and geometrical data of the model domain. Sketch of two opposite wires supplying the electric current.
5 CONCLUDING REMARKS

Many industrial processes in electrical engineering require the numerical solution of eddy current problems. In this paper, we have studied different finite element methods to approximate these problems in three-dimensional bounded domains. The finite element techniques
found in the literature are based either on potential formulations or on magnetic/electric field formulations. In fact, the potential formulations represent the basis of most commercial codes. We have reviewed the main features and shortcomings of the different techniques from modeling and numerical points of view. After a general overview, we have focused our attention on the analysis of finite element methods for electric or magnetic field formulations of eddy current problems, subject to realistic boundary conditions from the point of view of applications. We have presented the main results which arise from the mathematical analysis of these formulations; in some cases, boundary conditions which are neither essential nor natural have been imposed by means of Lagrange multipliers leading to mixed
formulations studied by classical techniques. Moreover, we have found that the Lagrange multipliers have a physical meaning which, depending on the unknown of the problem, correspond to either an electric surface potential or the input current intensities.

To numerically solve the proposed weak formulations, we have used edge elements to approximate the magnetic and electric fields. These elements only enforce the continuity of the tangential component and allow jumps on the normal components at material interfaces, which is an usual property of electromagnetic fields. Moreover, there are useful theoretical properties available concerning edge elements (see for instance [4, 10, 19, 61, 102]), which have allowed us to prove the reported convergence results for the discrete problems, under mild smoothness assumptions on the unknown fields.

We have used the proposed techniques to simulate the behavior of electrodes used in metallurgical electric furnaces for silicon and ferrosilicon production. We have shown a sampling of the numerical results we have obtained. These results have allowed us to explain quite satisfactorily the electrical behavior of the electrodes and have been used to optimize their design and working operation.

Although the numerical and mathematical analysis of eddy current problems has been an active research area during the last years in applied mathematics and electrical engineering, there are still many open problems. In next paragraphs we quote some lines of possible further research on this topic.

First, from a computational point of view, we notice that the discretization of the problems presented in Section 3 leads to large sparse linear systems of equations. This is a common feature of three-dimensional finite element problems. Thus, it is very important to choose an efficient solver, which due to the number of unknowns in many cases must be iterative. In this context, the efficient solution of linear systems arising from eddy current problems has been investigated during the last years. In particular, the use of multigrid techniques has been one of the main research directions. We refer the reader to Hiptmair [67] and references therein to see recent advances in this area for the discretization of the electric field formulation of the eddy current problem with essential boundary conditions.

An alternative approach has been developed by Perugia et al. [111, 112]. The algebraic linear systems arising in the discretization of the mixed problems presented in Section 3 have the typical common structure

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}
\]

Perugia et al. have investigated different iterative solvers based on this structure in the context of magnetostatic problems. The authors of the present paper used these ideas in eddy current problems and obtained promising results (see [119]), but without a rigorous algebraic analysis.

Another important aspect is the use of correctly refined meshes to numerically solve the problems with minimal computational effort. Beck et al. have investigated in [11] and [12] a posteriori error estimates for the eddy current problem with essential boundary conditions in terms of the electric field and the magnetic vector potential, respectively. It would be interesting to study similar techniques for other eddy current formulations.

On the other hand, numerical solution of ungauged vector potential formulations by using edge elements represent an interesting alternative. Thus, it would be interesting to compare in detail the efficiency between this and magnetic or electric field formulations. Moreover, to the best of the author’s knowledge, the mathematical and numerical analysis of ungauged potential formulations, has not been done yet.

Finally, coupling between eddy current models and others like thermal or hydrodynamic models still presents many open questions both from mathematical and numerical points of view.
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