Several finite element methods for the numerical computation of elastoacoustic vibrations are compared. They are applied to two formulations based on different variables to describe the fluid: pressure and displacement potential in one case, and displacements in the other. While the first one is discretized by standard Lagrangean finite elements for both variables, the second one is solved by “face” Raviart-Thomas elements. In each case we consider both tetrahedral and hexahedral meshes. Elastoacoustic eigenmodes have been computed for a test example by means of MATLAB implementations of all these methods. The numerical results allow us to compare all of them in terms of error versus number of degrees of freedom and computing time.

1. Introduction

During the last years a large amount of work have been devoted to the numerical computation of elastoacoustics vibration problems (see Bermúdez, Hervella and Rodríguez\(^1\) and the references therein). In the case of modal analysis of a coupled fluid-solid system, unsymmetric generalized eigenvalue problems arise when either pressure or displacement potential are used as the state variable for the fluid, which impede to use most standard eigensolvers. To overcome this drawback Morand and Ohayon\(^2\) have proposed to use both, pressure and displacement potential, to obtain a symmetric discrete problem.

An alternative approach to this pressure/potential formulation consists of using the displacement field to describe the fluid. Then, symmetrical eigenvalue problems are obtained but, in principle, two disadvantages arise: vector fields have to be handled and Lagrangean finite elements produce spurious eigenvalues (Kiefling and Feng\(^3\)). However, the latter can be avoided\(^1\) by using “face elements”, which are somehow similar to “edge elements” currently used in electromagnetism.

We consider here both formulations, discretized by means of tetrahedral and hexahe-
dral meshes. We compare the efficiencies of each approach in terms of accuracy versus computational cost.

2. Mathematical Modelling

Let us consider the harmonic vibrations of a (linear) elastic solid coupled with an inviscid, barotropic, compressible fluid (acoustic fluid).

We denote by \(\Omega_F\) and \(\Omega_S\) the three-dimensional interior and exterior domains, occupied by the fluid and the solid, respectively. The exterior boundary of \(\Omega_S\) is the union of two parts, \(\Gamma_D\) and \(\Gamma_N\), the structure being fixed on the former and free on the latter. We denote by \(\Gamma_I\) the interface between solid and fluid. Figure 1 shows corresponding two-dimensional domains for a better understanding of the notation.

![Fig. 1. Fluid and solid domains in 2D.](image)

The weak form of the pressure/potential formulation reads:\(^2\)

To find an eigenfrequency \(\omega\) and the corresponding mode shape given by \(\mathbf{W}\) (solid displacement field), \(P\) (fluid pressure) and \(\phi\) (fluid displacement potential), with \(\mathbf{W} = 0\) on \(\Gamma_D\), such that:

\[
\int_{\Omega_S} \sigma(\mathbf{W}) : \varepsilon(\mathbf{Z}) \, dx + \frac{1}{\rho_F c^2} \int_{\Omega_F} PQ \, dx = \omega^2 \left( \int_{\Omega_S} \rho_s \mathbf{W} \cdot \mathbf{Z} \, dx + \frac{1}{c^2} \int_{\Omega_F} \phi Q \, dx \right) + \frac{1}{c^2} \int_{\Gamma_F} P \psi \, dx - \int_{\Gamma_F} \rho_F \mathbf{v} \cdot \mathbf{v} \psi \, dx + \int_{\Gamma_I} \rho_F \mathbf{W} \cdot \mathbf{v} \psi \, dS + \int_{\Gamma_I} \rho_F \phi \mathbf{Z} \cdot \mathbf{v} \, dS, \tag{2.1}
\]

for all test functions \(\mathbf{Z}, Q\) and \(\psi\), with \(\mathbf{Z} = 0\) on \(\Gamma_D\).

On the other hand, for the displacement formulation, there holds:\(^1\)

To find an eigenfrequency \(\omega\) and the corresponding mode shape given by \(\mathbf{U}\) (fluid displacement field) and \(\mathbf{W}\) (solid displacement field), with \(\mathbf{U} \cdot \mathbf{v} = \mathbf{W} \cdot \mathbf{v}\) on \(\Gamma_I\) (interface kinematic constraint) and \(\mathbf{W} = 0\) on \(\Gamma_D\), such that:

\[
\int_{\Omega_S} \sigma(\mathbf{W}) : \varepsilon(\mathbf{Z}) \, dx + \int_{\Omega_F} \rho_F c^2 \text{div} \mathbf{U} \text{div} \mathbf{V} \, dx = \omega^2 \left( \int_{\Omega_S} \rho_s \mathbf{W} \cdot \mathbf{Z} \, dx + \int_{\Omega_F} \rho_F \mathbf{U} \cdot \mathbf{V} \, dx \right), \tag{2.2}
\]
for all test functions $\tilde{Y}$ and $\tilde{Z}$, with $\tilde{Y} \cdot \hat{v} = \tilde{Z} \cdot \hat{v}$ on $\Gamma_I$, and $\tilde{Z} = 0$ on $\Gamma_D$.

Notice that both formulations lead to symmetric eigenvalue problems.

3. Finite element computation: an hexahedral face element

We will compare the performances of different discretizations for the two eigenproblems above. In what follows we introduce one of them, based on hexahedral face elements, which has not been used before in the context of elastoacoustic problems.

Let us consider an hexahedral mesh $\mathcal{T}_h$ of $\Omega_F \cup \Omega_S$. We suppose that each element is completely included either in the solid or in the fluid, but compatibility on the fluid-solid interface is not required. Each component of the solid displacement is discretized by trilinear isoparametric finite elements (see for instance Petyt for details). This leads to 24 degrees of freedom per hexahedron.

Fluid displacements are discretized by the lowest order Raviart-Thomas isoparametric hexahedron. In order to describe the interpolating functions for this finite element, let us consider the Piola Transform $P_T$ associated with the trilinear map $F_T$ applying the reference hexahedron $\hat{T}$ (i.e., the unit cube) on a particular element $T$. $P_T$ allows us to transport vector fields from $T$ to $\hat{T}$ by $\hat{u} = \det(\nabla F_T)(\nabla F_T)^{-1}U$. Then the transformed interpolating function $\hat{u}$ must belong to the space of polynomial vector fields of the form: $\hat{u}(\hat{x}, \hat{y}, \hat{z}) = (a + b\hat{x}, c + d\hat{y}, e + f\hat{z})$. Thus, there exist six degrees of freedom per hexahedron which are the averaged values of the normal components of the displacement on the faces of the mesh. (See for instance Brezzi and Fortin’s book for further details.)

Finally, the interface kinematic condition is imposed in a weak sense by $\int_F (\hat{U} - \hat{W}) \cdot \hat{v} dS = 0$, for all the faces $F$ on the fluid-solid interface, leading to a nonconforming approximation. To do this, we perform a static condensation of the degrees of freedom of the fluid on the interface in terms of the nodal values of the solid displacements. So, we avoid introducing a
Lagrange multiplier to impose the kinematic constraint, thus reducing the dimension of the resulting matrices.

4. Numerical Results

In this section we show numerical results corresponding to the following finite element discretizations:

(i) pressure/potential formulation discretized by Lagrangean tetrahedral finite elements (degree 0 for pressure and degree 1 for potential);
(ii) pressure/potential formulation discretized by Lagrangean hexahedral finite elements (degree 0 for pressure and trilinear for potential);
(iii) displacement formulation discretized by lowest degree Raviart-Thomas tetrahedral finite elements;
(iv) displacement formulation discretized by lowest degree Raviart-Thomas hexahedral finite elements.

The test problem corresponds to water in a perfectly rigid cavity covered by a 3D clamped moderately thick steel plate (see Figure 4). The dimensions for the cavity are 4 m × 6 m × 1 m and for the plate 4 m × 6 m × 0.5 m. The physical data are \( \rho_v = 1000 \text{ kg/m}^3 \) and \( c = 1430 \text{ m/s} \) for the water, and \( \rho_s = 7700 \text{ kg/m}^3 \), Young modulus \( E = 1.44 \times 10^{11} \text{ Pa} \) and Poisson ratio \( \nu = 0.35 \) for the steel. We have considered a quarter of the geometry to reduce the number of degrees of freedom by imposing the symmetries of the different modes as constraints. As reference solution we have used the one obtained by using a plate model (Reissner-Mindlin)

for a fine mesh.

![Fig. 4. Geometry and mesh (h = 0.5; red: solid; blue: fluid).](image)

Tables 1 and 2 show the “exact” and computed first eigenfrequencies for the different methods indicated above, and for three successively refined meshes with degrees of refinement indicated by the edge length \( h \) of each cubic element.

Figures 5 to 8 show the error versus either the number of degrees of freedom (DOF) or the computer time, required to calculate the ten first eigenvalues by using Version 5.3 of MATLAB eigensolver \texttt{eigs}. This eigensolver computes the eigenvalues and eigenvectors of
Table 1. Eigenfrequencies computed by Lagrangean elements applied to the pressure/potential formulation.

<table>
<thead>
<tr>
<th>Mode</th>
<th>h = 0.5</th>
<th>h = 0.5/2</th>
<th>h = 0.5/3</th>
<th>h = 0.5</th>
<th>h = 0.5/2</th>
<th>h = 0.5/3</th>
<th>“Exact”</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_{010}$</td>
<td>735.296</td>
<td>723.268</td>
<td>715.669</td>
<td>721.379</td>
<td>708.879</td>
<td>704.384</td>
<td>697.555</td>
</tr>
<tr>
<td>$\omega_{F00}$</td>
<td>1117.210</td>
<td>1101.875</td>
<td>1095.428</td>
<td>1113.917</td>
<td>1095.135</td>
<td>1089.600</td>
<td>1081.911</td>
</tr>
<tr>
<td>$\omega_{S1}$</td>
<td>1446.008</td>
<td>1280.564</td>
<td>1170.654</td>
<td>1231.465</td>
<td>1096.456</td>
<td>1059.475</td>
<td>1014.635</td>
</tr>
</tbody>
</table>

Table 2. Eigenfrequencies computed by Raviart-Thomas elements applied to the displacement formulation.

<table>
<thead>
<tr>
<th>Mode</th>
<th>h = 0.5</th>
<th>h = 0.5/2</th>
<th>h = 0.5/3</th>
<th>h = 0.5</th>
<th>h = 0.5/2</th>
<th>h = 0.5/3</th>
<th>“Exact”</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_{010}$</td>
<td>731.289</td>
<td>722.231</td>
<td>719.657</td>
<td>708.484</td>
<td>704.211</td>
<td>697.555</td>
<td></td>
</tr>
<tr>
<td>$\omega_{F00}$</td>
<td>1104.640</td>
<td>1098.729</td>
<td>1107.276</td>
<td>1093.580</td>
<td>1088.925</td>
<td>1081.911</td>
<td></td>
</tr>
<tr>
<td>$\omega_{S1}$</td>
<td>1416.978</td>
<td>1275.222</td>
<td>1228.700</td>
<td>1096.031</td>
<td>1059.303</td>
<td>1014.635</td>
<td></td>
</tr>
</tbody>
</table>

a generalized eigenvalue problem of the form $Ax = \lambda Bx$ with $B$ symmetric and positive definite. For the displacement formulation we obtain a discretized problem of this form by eliminating the degrees of freedom of the fluid on the interface in terms of those of the solid, by means of the kinematic constraint. So `eigs` can be directly applied.

This is not the case, however, for the pressure/potential formulation and then some previous transformations in the discrete problem have to be made. More precisely, we have rewritten the matrix eigenproblem $Bx = \frac{1}{\lambda - \delta}(A - \delta B)x$. If $\delta > 0$ is small enough (i.e., $0 < \delta < \min\{\lambda_1^s, \lambda_1^f\}$, with $\lambda_1^s$ and $\lambda_1^f$ being the smallest eigenvalues of the uncoupled vibration problems for the solid and the fluid, respectively), then $A - \delta B$ is semipositive definite. If symmetry conditions are imposed, this matrix turns out to be positive definite. Otherwise, a very small perturbation $\epsilon I$ ($0 < \epsilon \ll \delta$) should be still added, in order to be able to use MATLAB eigen solver `eigs`.

![Fig. 5. Error versus number of DOF (log-log scale)](image1)

![Fig. 6. Error versus computer time (log-log scale)](image2)
By observing these figures we may conclude that both formulations involve similar computational costs. However, in both cases, hexahedra are much less expensive than tetrahedra.

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