Finite element approximation of a displacement formulation for time-domain elastoacoustic vibrations

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Abstract

This paper deals with the numerical solution of a system of second-order in time partial differential equations modeling the vibrations of a coupling between an elastic solid and an inviscid compressible fluid. Both media are described in terms of their respective displacement fields. The problem is solved in the time domain by combining a Newmark’s scheme for time discretization with a finite element method for space discretization. The latter combines standard Lagrangian elements in the solid with lowest-order Raviart-Thomas elements in the fluid. Stability is proved and numerical results showing the good behavior of the method are reported.

Key words: Fluid-structure interaction, time-domain elastoacoustics, finite element methods

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1 Introduction

The structural acoustics (or elastoacoustics) problem concerns the coupled vibrations of an elastic structure and an acoustic fluid (i.e. an inviscid compressible barotropic fluid). Numerical simulation of these vibrations is of primary interest in acoustical engineering, particularly in studies aiming to decrease noise by either passive or active control methods. Most of the bibliography on the subject concerns harmonic vibrations and hence eigenvalue problems (see for instance the book by Morand and Ohayon [8] and references therein). Regarding the time-domain problem, let us mention [3] and [9].

Several models can be used depending on the field which is taken to describe the motion of the fluid. Indeed, one can use pressure, displacement, displacement potential, velocity, velocity potential, or some combinations of them. In the present paper we use the displacement field. The main advantages of this choice are: first, this field is the same as the one usually adopted to describe the solid, so interface conditions are very easy to write and handle, and second, the obtained weak formulation of the coupled problem is symmetric, a property which is inherited by the discrete problem and makes it easier to solve.

It is well known (see [6]) that, for the corresponding spectral problems, numerical approximation of the fluid displacements by using standard Lagrangian continuous piecewise linear finite elements leads to spurious eigenmodes, which are very difficult to identify because they are placed among the physical ones. This is why in [2] (see also [1]) an alternative approach has been proposed: to use Raviart-Thomas finite elements instead of the above mentioned classical ones. The advantage of them is that, unlike the Lagrangian finite elements, the corresponding discrete spaces contain subspaces which approximate in a suitable way the infinite-dimensional subspace of rotational modes (i.e. those corresponding to the null vibration frequency). More precisely, the zero-frequency modes of the continuous problem are correctly approximated by the zero-frequency modes of the discrete one.

In the present paper we study a numerical approximation to solve the pure displacement formulation of the elastoacoustics problem in the time domain by using the above finite element approximation. We conclude that the good behavior exhibited by this method for spectral problems is also observed for the time-domain transient problem. In particular, the solutions corresponding to initial rotational modes are well approximated.

The outline of the paper is as follows. In Section 2 we introduce the model and obtain a weak formulation. Existence and uniqueness of solution can be readily proved by applying well-known results for abstract second-order in
time evolution equations. Section 3 concerns numerical approximation. First, we introduce a finite element semi-discretization in space by combining Lagrangian finite elements in the solid and Raviart-Thomas elements in the fluid, with transmission conditions across the fluid-solid interface imposed in a weak sense. Then we apply Newmark’s finite difference scheme for time discretization. We end this section by proving the stability of the proposed scheme. A matrix description of the algorithm, including static condensation to impose the interface constraint, is given in Section 4. Finally, in Section 5, we present the numerical results obtained for some test examples, which show the good behavior of the developed numerical method.

2 Problem statement

Let us consider as a model problem, an elastic (two-dimensional) vessel enclosing an inviscid compressible fluid (see Figure 1). The domains occupied by the fluid and the solid are denoted by \( \Omega_F \) and \( \Omega_S \), respectively. Let \( \Gamma_I \) be the interface between both media and \( \nu \) its normal unit vector pointing outwards \( \Omega_F \). The external boundary of the solid is the union of two parts, \( \Gamma_D \) and \( \Gamma_N \), the structure being fixed along the former. Finally \( n \) denotes the unit outward normal vector along \( \Gamma_N \).

![Diagram of fluid and solid domains](image)

We use the following notations for the physical magnitudes in the fluid:

- \( \mathbf{u}^F \): the displacement vector,
- \( c \): the sound velocity,
- \( \rho_F \): the density,

and in the solid:
• $u^s$: the displacement vector,
• $\rho^s$: the density,
• $\lambda^s$ and $\mu^s$: the Lamé coefficients,
• $\varepsilon(u^s)$: the strain tensor defined by
  \[ \varepsilon_{ij} \overset{\text{def}}{=} \frac{1}{2} \left( \frac{\partial u^s_i}{\partial x_j} + \frac{\partial u^s_j}{\partial x_i} \right), \]
  $i, j = 1, 2$,
• $\sigma(u^s)$: the stress tensor, which we assume related to the strains by Hooke’s law, i.e.,
  \[ \sigma_{ij} = \lambda^s \sum_{k=1}^2 \varepsilon_{kk} \delta_{ij} + 2\mu^s \varepsilon_{ij}, \]
  $i, j = 1, 2$.

If a time-dependent load $G(t)$ is applied on $\Gamma_N$, the equations governing the motion of the coupled system are (see for instance [8]):

\[
\begin{align*}
\rho^s \frac{\partial^2 u^s}{\partial t^2} - \text{div} \sigma(u^s) &= 0 \quad \text{in } \Omega_s, \quad \text{(2.1)} \\
\rho^\nu \frac{\partial^2 u^\nu}{\partial t^2} - \nabla (\rho^\nu c^2 \text{div } u^\nu) &= 0 \quad \text{in } \Omega_\nu, \quad \text{(2.2)} \\
\quad u^s \cdot \nu = u^\nu \cdot \nu \quad \text{on } \Gamma_1, \quad \text{(2.3)} \\
\quad \sigma(u^s) \nu = \rho^\nu c^2 \text{div } u^\nu \nu \quad \text{on } \Gamma_1, \quad \text{(2.4)} \\
\quad \sigma(u^s) n = G \quad \text{on } \Gamma_N, \quad \text{(2.5)} \\
\quad u^s = 0 \quad \text{on } \Gamma_D, \quad \text{(2.6)} 
\end{align*}
\]

plus initial conditions

\[
\begin{align*}
\quad u^s(0) &= u^s_0, \quad \quad \quad \quad \quad u^\nu(0) = u^\nu_0, \quad \quad \frac{\partial u^s}{\partial t}(0) = u^s_1, \quad \quad \text{and} \quad \quad \frac{\partial u^\nu}{\partial t}(0) = u^\nu_1. \quad \text{(2.7)}
\end{align*}
\]

Throughout this paper we use standard notation for Sobolev spaces, norms, and seminorms. We also denote by $H^1_{\Gamma_D}(\Omega_s)^2$ the closed subspace of functions in $H^1(\Omega_s)^2$ vanishing on $\Gamma_D$, and $H(\text{div}, \Omega_\nu)^2 \overset{\text{def}}{=} \{ u \in L^2(\Omega_\nu)^2 : \text{ div } u \in L^2(\Omega_\nu) \}$, endowed with the norm defined by

\[
\| u \|_{H(\text{div}, \Omega_\nu)^2} \overset{\text{def}}{=} \| \text{ div } u \|_{L^2(\Omega_\nu)^2} + \| u \|_{L^2(\Omega_\nu)^2}.
\]

Let us define the Hilbert spaces

\[ H \overset{\text{def}}{=} L^2(\Omega_s)^2 \times L^2(\Omega_\nu)^2 \quad \text{and} \quad X \overset{\text{def}}{=} H^1_{\Gamma_D}(\Omega_s)^2 \times H(\text{div}, \Omega_\nu), \]

endowed with the corresponding product norms $| \cdot |$ and $\| \cdot \|$, respectively, and

\[ V \overset{\text{def}}{=} \{ (u^s, u^\nu) \in X : u^s \cdot \nu = u^\nu \cdot \nu \text{ on } \Gamma_1 \}, \]

which is a closed subspace of $X$. Both, $V$ and $X$, are densely included in $H$ but the inclusions are not compact. By identifying $H$ with its topological dual and denoting by $V'$ and $X'$ the dual spaces of $V$ and $X$, respectively, we have

\[ V \hookrightarrow H \hookrightarrow V' \quad \text{and} \quad X \hookrightarrow H \hookrightarrow X'. \]
all the inclusions being dense. We denote, in both cases, \( \| \cdot \| \ast \) the corresponding dual norm and \( \langle \cdot , \cdot \rangle \) the duality pairing.

2.1 Variational formulation

Multiplying (2.1) by a test function \( v^S \in H^1_D(\Omega_S)^2 \) and applying a Green’s formula, we have

\[
\int_{\Omega_S} \rho_s \frac{\partial^2 u^S}{\partial t^2} \cdot v^S + \int_{\Omega_S} \sigma(u^S) : \varepsilon(v^S) + \int_{\Gamma_i} \sigma(u^S) \nu \cdot v^S \, d\Gamma - \int_{\Gamma_N} \sigma(u^S) n \cdot v^S \, d\Gamma = 0.
\]

Similarly, multiplying (2.2) by a test function \( v^F \in H(\text{div}, \Omega_F) \) and applying a Green’s formula we obtain

\[
\int_{\Omega_F} \rho_F \frac{\partial^2 u^F}{\partial t^2} \cdot v^F + \int_{\Omega_F} \rho_F c^2 \text{div} u^F \text{div} v^F - \int_{\Gamma_i} \rho_F c^2 \text{div} u^F \cdot v^F \, d\Gamma = 0.
\]

Adding this two equations and taking into account the boundary and interface conditions (2.4)-(2.5) and that \( (v^S, v^F) \) satisfies the kinematic constraint (i.e., \( v^S \cdot \nu = v^F \cdot \nu \)), our problem can be written in a weak sense as follows:

**Problem 1** Given \( (u^S_0, u^F_0) \in V, \ (u^S_1, u^F_1) \in H^1_D(\Omega_S)^2 \) and \( G \in L^2(\Gamma_N)^2 \), find \( u^S : [0, T] \to H^1_D(\Omega_S)^2 \) and \( u^F : [0, T] \to H(\text{div}, \Omega_F) \) such that, \( \forall t \in (0, T) \), \((u^S(t), u^F(t)) \in V,

\[
\int_{\Omega_S} \rho_s \frac{\partial^2 u^S}{\partial t^2} \cdot v^S + \int_{\Omega_F} \rho_F \frac{\partial^2 u^F}{\partial t^2} \cdot v^F + \int_{\Omega_S} \sigma(u^S) : \varepsilon(v^S) + \int_{\Omega_F} \rho_F c^2 \text{div} u^F \text{div} v^F = \int_{\Gamma_N} G \cdot v^S \, d\Gamma \quad \forall (v^S, v^F) \in V,
\]

and

\[
u^S(0) = u^S_0, \quad u^F(0) = u^F_0, \quad \frac{\partial u^S}{\partial t}(0) = u^S_1, \quad \text{and} \quad \frac{\partial u^F}{\partial t}(0) = u^F_1.
\]

Let us remark that, if \( G(t) \notin L^2(\Gamma_N)^2 \), the term \( \int_{\Gamma_N} G \cdot v^S \, d\Gamma \) in the equation above must be understood as the duality pairing \( \langle G(t), v^S \rangle_{H^{-\frac{1}{2}}(\Gamma_N)^2 \times H^{\frac{1}{2}}(\Gamma_N)^2} \).

We define the bilinear symmetric form \( a \) by

\[
a ((u^S, u^F), (v^S, v^F)) \overset{\text{def}}{=} \int_{\Omega_S} \sigma(u^S) : \varepsilon(v^S) + \int_{\Omega_F} \rho_F c^2 \text{div} u^F \text{div} v^F,
\]
and the linear functional \( g(t) \in X' \) by
\[
\langle g(t), (v^s, v^f) \rangle \overset{\text{def}}{=} \int_{r_N} G(t) \cdot v^s \, d\Gamma \quad \forall (v^s, v^f) \in X.
\]

From now on we denote \( u = (u^s, u^f) \). We also define the following weighted inner product in \( H \)
\[
(u, v)_\rho \overset{\text{def}}{=} \int_{\Omega_s} \rho^s u^s \cdot v^s + \int_{\Omega_f} \rho^f u^f \cdot v^f
\]
and its associated norm \( |u|_\rho \overset{\text{def}}{=} (u, u)_\rho^{1/2} \). Clearly, this weighted norm is equivalent to \( | \cdot | \). Finally, we denote \( \langle \cdot, \cdot \rangle_\rho \) the duality pairing between \( V' \) and \( V \) weighted with this densities (i.e., associated with the inclusions \( (V, \| \cdot \|) \hookrightarrow (H, | \cdot |) \hookrightarrow (V', \| \cdot \|_*) \)).

It is simple to show that the bilinear form \( a \) satisfies a Gårding’s inequality; i.e., there exist \( \gamma \in \mathbb{R} \) and \( \alpha > 0 \) such that
\[
a(u, u) + \gamma |u|_\rho^2 \geq \alpha \|u\|^2 \quad \forall u \in X. \tag{2.7}
\]

2.2 Existence and uniqueness of solution

Let us make precise the functional spaces where the solution of problem 1 should lie and settle the existence and uniqueness of solution. First, we rewrite this problem as follows:

**Problem 2** Given \( u_0 \in V \), \( u_1 \in H \) and \( g \in L^2(0, T; X') \), find \( u : [0, T] \rightarrow V \) satisfying
\[
\left\langle \frac{d^2 u}{dt^2} (\cdot), v \right\rangle_\rho + a (u(\cdot), v) = \langle g(\cdot), v \rangle \quad \forall v \in V, \quad \text{in } \mathcal{D}'([0, T]),
\]
and the initial conditions
\[
u(0) = u_0 \quad \text{and} \quad \frac{du}{dt}(0) = u_1.
\]

Since the linear functional \( g \) involves a boundary term, it does not belong to the space \( L^2(0, T; H) \). Therefore, we cannot apply the standard existence results (see, for instance, [4] or [7]). However, an extension of these included in [5] valid for more regular data \( g \) can be readily adapted to our problem allowing us to prove the following result:
Theorem 1  If \( \frac{dg}{dt} \in L^2(0, T; X') \) then, there exists a unique solution of problem 2 satisfying

\[
\mathbf{u} \in L^\infty(0, T; V) \quad \text{and} \quad \frac{d\mathbf{u}}{dt} \in L^\infty(0, T; H).
\]

3 Numerical approximation

3.1 Space discretization: the finite element method.

The equations for solid and fluid involve different differential operators and functional spaces; then, it makes sense to use different type of finite elements for each of them to discretize the variational problem 1. For the corresponding eigenvalue problem, that is, the problem of determining the free harmonic vibration of the system, it is well known that the use of a displacement formulation to describe the state of the fluid discretized with classical Lagrangian finite elements leads to the presence of spurious or circulation modes. A discretization avoiding spurious modes typical of displacement formulations and leading to optimal order computation of eigenvalues and eigenfunctions has been introduced and analyzed in [1,2]. In this section we show that the same discretization can be used for the time-domain problem.

We consider two regular triangulations of \( \Omega_F \) and \( \Omega_S \), such that the end points of \( \Gamma_D \) and \( \Gamma_N \) coincide with nodes of the mesh in \( \Omega_S \). Compatibility of these meshes on the fluid-solid interface is not needed, but simplifies the calculus.

We use standard three-node elements (i.e., piecewise linear and continuous) for the displacement of the solid \( \mathbf{u}_S^h \), whereas for the fluid displacements \( \mathbf{u}_F^h \) we use lowest-order Raviart-Thomas elements (see [10]).

The kinematic constraint (2.3) should be imposed somehow to the discrete displacements. Since doing it strongly (i.e., \( \mathbf{u}_F^h \cdot \mathbf{n} = \mathbf{u}_S^h \cdot \mathbf{n} \) on \( \Gamma_I \)) is too stringent (see [2]), we impose it in the following weak way:

\[
\int_{\ell} \mathbf{u}_F^h \cdot \mathbf{n} \, d\Gamma = \int_{\ell} \mathbf{u}_S^h \cdot \mathbf{n} \, d\Gamma \quad \forall \text{ fluid mesh edge } \ell \subset \Gamma_I. \tag{3.1}
\]

This kinematic constraint on the interface allows eliminating by static condensation the degrees of freedom of the fluid displacement corresponding to these edges. The discrete analogue of \( V \) is then

\[
V_h \overset{\text{def}}{=} \{(\mathbf{u}_S^h, \mathbf{u}_F^h) \text{ satisfying (3.1) and } \mathbf{u}_h^S = 0 \text{ on } \Gamma_D\}.
\]
Thus we are led to the following semi-discrete problem:

**Problem 3** Given $u_{0h}$ and $u_{1h}$, approximations in $V_h$ of $u_0$ and $u_1$, respectively, and $G \in L^2\left(0, T; H^{-\frac{1}{2}}(\Gamma_N)^2\right)$, find $(u^s_h, u^F_h) \in L^2(0,T,V_h)$ such that

$$
\int_{\Omega_S} \rho_s \frac{\partial^2 u^s_h}{\partial t^2} \cdot v_h^s + \int_{\Omega_F} \rho_F \frac{\partial^2 u^F_h}{\partial t^2} \cdot v_h^F + \int_{\Omega_S} \sigma(u^s_h) : \varepsilon(v^s_h) + \int_{\Omega_F} \rho_F c^2 \text{div} u^F_h \text{div} v^F_h
$$

$$= \int_{\Gamma_N} G \cdot v^s_h \, d\Gamma \quad \forall (v^s_h, v^F_h) \in V_h, \quad \text{in } D'(0,T]
$$

and

$$u_h(0) = u_{0h}, \quad \frac{du_h}{dt}(0) = u_{1h}.$$ 

Since in general $u^F_h \cdot \nu \neq u^s_h \cdot \nu$ on $\Gamma_I$, then $V_h \not\subset V$, and hence the problem above is a non-conforming approximation of problem 1.

3.2 Time discretization: Newmark’s method

For the time discretization of the differential system in problem 3, we will apply Newmark’s method. This is a method of order 2 with respect to the time step (see for instance [11]) and it is very much used to solve second-order in time equations.

We introduce a discretization of the time interval $[0,T]$ with step size $\Delta t = \frac{T}{N}$:

$$t_n = n\Delta t, \quad n = 0, \ldots, N.$$ 

If $G$ is a regular enough function with respect to $t$, we denote $g_n \overset{\text{def}}{=} g(t_n) \in X'$, for $n = 0, \ldots, N$. We also denote $u^s_h$ the approximation of $u_h(t_n)$ that we are going to obtain with Newmark’s method. Then, given the initial data $u_{0h}$ and $u_{1h}$ in $V_h$ (approximations of $u_0$ and $u_1$, respectively), the algorithm of Newmark’s method applied to our problem can be written as follows:

**Algorithm 1** For $u_{0h}$, $u_{1h} \in V_h$ and $g(t_n) \in X'$, $n = 0, \ldots, N$, being given data, let:

- $u^0_h = u_{0h}$,
- $u^1_h \in V_h$ be the solution of

$$
\left( \frac{u^1_h - u^0_h - \Delta t u_{1,h}}{\Delta t^2}, v_h \right)_\rho + \frac{1}{4} a \left( u^1_h + u^0_h, v_h \right) = \frac{1}{4} \langle g_1 + g_0, v_h \rangle \quad \forall v_h \in V_h,
$$

where $a$ is the a priori given constant.
• and, for \( n = 1, \ldots, N - 1 \),

\[
\mathbf{u}_h^{n+1} \in V_h \text{ be the solution of }
\]
\[
\left( \frac{\mathbf{u}_h^{n+1} - 2 \mathbf{u}_h^n + \mathbf{u}_h^{n-1}}{\Delta t^2}, \mathbf{v}_h \right) + \frac{1}{4} \rho \left( \mathbf{u}_h^{n+1} + 2 \mathbf{u}_h^n + \mathbf{u}_h^{n-1}, \mathbf{v}_h \right) = \frac{1}{4} \left( g_{n+1} + 2 g_n + g_{n-1}, \mathbf{v}_h \right) \quad \forall \mathbf{v}_h \in V_h.
\]

It is simple to show that this algorithm is well-posed for any \( \Delta t > 0 \) (see Remark 2 below for a matrix description of the algorithm from which this can be easily verified). Consequently, we have existence and uniqueness of the discrete solution.

### 3.3 Stability

We start the proof of the stability of the above scheme by proving the following lemmas:

**Lemma 1** The following inequality holds \( \forall n = 0, \ldots, N - 1 \):

\[
\left| \mathbf{u}_h^{n+1} + \mathbf{u}_h^n \right|_{\rho}^2 \leq 12 | \mathbf{u}_{0,h} |_{\rho}^2 + 12 T \Delta t \sum_{k=0}^{n-1} \left| \frac{\mathbf{u}_h^{k+1} - \mathbf{u}_h^k}{\Delta t} \right|_{\rho}^2 + 3 (\Delta t)^2 \left| \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} \right|_{\rho}^2.
\]

**PROOF.** We have

\[
\left| \mathbf{u}_h^{n+1} + \mathbf{u}_h^n \right|_{\rho}^2 = 2 | \mathbf{u}_{0,h} |_{\rho}^2 + 2 \Delta t \sum_{k=0}^{n-1} \left| \frac{\mathbf{u}_h^{k+1} - \mathbf{u}_h^k}{\Delta t} + \Delta t \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} \right|_{\rho}^2,
\]

\[
\leq 12 | \mathbf{u}_{0,h} |_{\rho}^2 + 12 (\Delta t)^2 \sum_{k=0}^{n-1} \left| \frac{\mathbf{u}_h^{k+1} - \mathbf{u}_h^k}{\Delta t} \right|_{\rho}^2 + 3 (\Delta t)^2 \left| \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} \right|_{\rho}^2
\]

\[
\leq 12 | \mathbf{u}_{0,h} |_{\rho}^2 + 12 T \Delta t \sum_{k=0}^{n-1} \left| \frac{\mathbf{u}_h^{k+1} - \mathbf{u}_h^k}{\Delta t} \right|_{\rho}^2 + 3 (\Delta t)^2 \left| \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} \right|_{\rho}^2.
\]

Therefore, we conclude the lemma. \( \square \)

**Lemma 2** If \( g \in L^\infty(0,T;X') \), then there exists \( C_1 > 0 \) such that, for any time step \( \Delta t < \sqrt{\frac{4}{3\gamma}} \) with \( \gamma \) being the constant in Gårding’s inequality (2.7), there holds:

\[
\| \mathbf{u}_h^1 + \mathbf{u}_h^0 \| \leq C_1 \quad \text{and} \quad \left| \frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\Delta t} \right|_{\rho} \leq C_1.
\]
**Lemma 3**

If \( g \in L^\infty(0, T; X') \) and \( \frac{dg}{dt} \in L^2(0, T; X') \), then, \( \forall \epsilon > 0, \exists C_\epsilon > 0 \) independent of \( \Delta t \) such that

\[
\left| \sum_{k=1}^{n} \Delta t \left( g_{k+1} + 2g_k + g_{k-1}, \frac{u_{h}^{k+1} - u_{h}^{k-1}}{\Delta t} \right) \right| \leq C_\epsilon + \epsilon \left\| u_{h}^{n+1} + u_{h}^{n} \right\|^2 + \frac{\Delta t}{2} \sum_{k=1}^{n-1} \left\| u_{h}^{k+1} + u_{h}^{k} \right\|^2 + \left\| u_{h}^{1} + u_{h}^{0} \right\|^2.
\]

**Proof.** It is simple to show that

\[
\left| \frac{u_{h}^{1} - u_{h}^{0}}{\Delta t} \right|^2 + \frac{1}{4} \left( u_{h}^{1} + u_{h}^{0}, u_{h}^{1} + u_{h}^{0} \right) = \frac{1}{4} \langle g_1 + g_0, u_{h}^{1} + u_{h}^{0} \rangle - \frac{1}{2} \langle g_1 + g_0, u_{0,h} \rangle + \frac{1}{2} \left( \frac{u_{1,h}}{\Delta t}, \frac{u_{1,h} - u_{h}^{0}}{\Delta t} \right) + \frac{1}{2} \left( u_{1,h}, \frac{u_{1,h} - u_{h}^{0}}{\Delta t} \right) + \frac{1}{2} \left( u_{1,h}, \frac{u_{1,h} - u_{h}^{0}}{\Delta t} \right).
\]

Then, applying Gárding’s inequality (2.7) and the fact that the bilinear form \( a \) is bounded on \( X \) (i.e., there exists a positive constant \( \|a\| \) such that \( |a(u, v)| \leq \|a\| \|u\| \|v\| \forall u, v \in X \)), we have

\[
\left| \frac{u_{h}^{1} - u_{h}^{0}}{\Delta t} \right|^2 + \frac{\alpha}{4} \left\| u_{h}^{1} + u_{h}^{0} \right\|^2 \leq \frac{1}{4} \left\| g_1 + g_0 \right\| \left\| u_{h}^{1} + u_{h}^{0} \right\| + \frac{1}{2} \left\| g_1 + g_0 \right\| \left\| u_{0,h} \right\| + \left| u_{1,h} \right| \left\| \frac{u_{1,h} - u_{h}^{0}}{\Delta t} \right\| + \frac{3\gamma (\Delta t)^2}{4} \left\| \frac{u_{1,h} - u_{h}^{0}}{\Delta t} \right\|^2 + 3\gamma \left\| u_{0,h} \right\|^2.
\]

where we have used the previous lemma with \( n = 0 \) for the last inequality. Hence, for \( 3\gamma (\Delta t)^2 < 4 \), straightforward computations allow us to conclude the lemma. \( \square \)
\[
\sum_{k=1}^{n} \Delta t \left\langle g_{k+1} + 2g_k + g_{k-1}, \frac{u_h^{k+1} - u_h^{k-1}}{\Delta t} \right\rangle \\
= -\sum_{k=1}^{n-1} \Delta t \left\langle g_{k+1} + 2g_k + g_{k-1}, \frac{g_{k+1} - g_{k-1}}{\Delta t}, u_h^{k+1} + u_h^k \right\rangle \\
+ \left\langle g_{n+1} + 2g_n + g_{n-1}, u_h^{n+1} + u_h^n \right\rangle - \left\langle g_2 + 2g_1 + g_0, u_h^1 + u_h^0 \right\rangle.
\]

The first term in the right hand side above can be bounded by

\[
\left| \sum_{k=1}^{n-1} \Delta t \left\langle g_{k+1} + 2g_k + g_{k-1}, \frac{g_{k+1} - g_{k-1}}{\Delta t}, u_h^{k+1} + u_h^k \right\rangle \right| \\
\leq \frac{1}{2} \sum_{k=1}^{n-1} \Delta t \left\| \frac{g_{k+1} - g_{k-1}}{\Delta t} \right\|_\infty^2 + \frac{1}{2} \sum_{k=1}^{n-1} \Delta t \left\| u_h^{k+1} + u_h^k \right\|^2.
\]

Now, taking into consideration that

\[
g_{k+2} + g_{k+1} - g_k - g_{k-1} = \int_{t_k}^{t_{k+1}} \frac{dg}{dt}(\tau) d\tau + \int_{t_{k-1}}^{t_k} \frac{dg}{dt}(\tau) d\tau,
\]

it is simple to obtain the estimate

\[
\frac{1}{2} \sum_{k=1}^{n-1} \Delta t \left\| \frac{g_{k+1} - g_{k-1}}{\Delta t} \right\|_\infty^2 \leq \left\| \frac{dg}{dt} \right\|_{L^2(0,T;X')}^2.
\]

On the other hand, \( \forall \epsilon > 0 \) we have

\[
\left| \left\langle g_{n+1} + 2g_n + g_{n-1}, u_h^{n+1} + u_h^n \right\rangle \right| \leq \frac{1}{4\epsilon} \left\| g_{n+1} + 2g_n + g_{n-1} \right\|_\infty^2 + \epsilon \left\| u_h^{n+1} + u_h^n \right\|^2 \\
\leq \frac{4}{\epsilon} \left\| g \right\|_{L^\infty(0,T;X')}^2 + \epsilon \left\| u_h^{n+1} + u_h^n \right\|^2
\]

and, similarly,

\[
\left| \left\langle g_2 + 2g_1 + g_0, u_h^1 + u_h^0 \right\rangle \right| \leq 4 \left\| g \right\|_{L^\infty(0,T;X')}^2 + \left\| u_h^1 + u_h^0 \right\|^2.
\]

Hence, we conclude the lemma. \( \Box \)

**Theorem 2 (Stability)** If \( g \in L^\infty(0,T;X') \) and \( \frac{dg}{dt} \in L^2(0,T;X') \), then \( \exists C > 0 \) such that, for any time step \( \Delta t < \sqrt{\frac{1}{3\gamma}} \) with \( \gamma \) being the constant in Gårding’s inequality (2.7), there holds

\[
\left\| u_h^{n+1} + u_h^n \right\| \leq C \quad \text{and} \quad \left\| \frac{u_h^{n+1} - u_h^n}{\Delta t} \right\|_\rho \leq C, \quad n = 0, \ldots, N - 1.
\]
PROOF. First, for \( k = 1, \ldots, N - 1 \), we have

\[
\left( \frac{u_{h}^{k+1} - 2u_{h}^{k} + u_{h}^{k-1}}{\Delta t^2}, u_{h}^{k+1} - u_{h}^{k-1} \right) = \left| \frac{u_{h}^{k+1} - u_{h}^{k}}{\Delta t} \right|_{\rho}^2 - \left| \frac{u_{h}^{k} - u_{h}^{k-1}}{\Delta t} \right|_{\rho}^2
\]

and

\[
a \left( u_{h}^{k+1} + 2u_{h}^{k} + u_{h}^{k-1}, u_{h}^{k+1} - u_{h}^{k-1} \right)
= a \left( u_{h}^{k+1} + u_{h}^{k}, u_{h}^{k+1} + u_{h}^{k} \right) - a \left( u_{h}^{k} + u_{h}^{k-1}, u_{h}^{k} + u_{h}^{k-1} \right).
\]

Then, by applying the second equation in algorithm 1 with \( n = k \) and \( v_{n} = u_{h}^{k+1} - u_{h}^{k-1} \), we obtain

\[
\left| \frac{u_{h}^{k} - u_{h}^{k-1}}{\Delta t} \right|_{\rho}^2 + \frac{1}{4} a \left( u_{h}^{k+1} + u_{h}^{k}, u_{h}^{k+1} + u_{h}^{k} \right)
= \left| \frac{u_{h}^{k} - u_{h}^{k-1}}{\Delta t} \right|_{\rho}^2 + \frac{1}{4} a \left( u_{h}^{k} + u_{h}^{k-1}, u_{h}^{k} + u_{h}^{k-1} \right)
+ \frac{1}{4} \left( g_{k+1} + 2g_{k} + g_{k-1}, u_{h}^{k+1} - u_{h}^{k-1} \right).
\]

Adding this equality for \( k = 1 \) to \( n \) leads to

\[
\left| \frac{u_{h}^{n+1} - u_{h}^{n}}{\Delta t} \right|_{\rho}^2 + \frac{1}{4} a \left( u_{h}^{n+1} + u_{h}^{n}, u_{h}^{n+1} + u_{h}^{n} \right)
= \left| \frac{u_{h}^{1} - u_{h}^{0}}{\Delta t} \right|_{\rho}^2 + \frac{1}{4} a \left( u_{h}^{1} + u_{h}^{0}, u_{h}^{1} + u_{h}^{0} \right)
+ \frac{1}{4} \sum_{k=1}^{n} \Delta t \left( g_{k+1} + 2g_{k} + g_{k-1}, \frac{u_{h}^{k+1} - u_{h}^{k-1}}{\Delta t} \right).
\]

Hence, by using Lemmas 2 and 3, we obtain \( \forall \rho > 0 \)

\[
\left| \frac{u_{h}^{n+1} - u_{h}^{n}}{\Delta t} \right|_{\rho}^2 + \frac{1}{4} a \left( u_{h}^{n+1} + u_{h}^{n}, u_{h}^{n+1} + u_{h}^{n} \right)
\leq C_{\epsilon} + \frac{\epsilon}{4} \left\| u_{h}^{n+1} + u_{h}^{n} \right\|^2 + \frac{\Delta t}{8} \sum_{k=1}^{n-1} \left\| u_{h}^{k+1} + u_{h}^{k} \right\|^2,
\]

with \( C_{\epsilon} \) depending on \( \epsilon \). Then, by Gårding’s inequality (2.7), we have
\[
\frac{u^{n+1}_h - u^n_h}{\Delta t} \rho + \frac{\alpha}{4} ||u^{n+1}_h + u^n_h||^2 \\
\leq C' + \frac{\epsilon}{4} ||u^{n+1}_h + u^n_h||^2 + \frac{\Delta t}{8} \sum_{k=1}^{n-1} ||u^{k+1}_h + u^k_h||^2 + \frac{\gamma}{4} ||u^{n+1}_h + u^n_h||^2
\]

and, hence, by Lemma 1,

\[
\frac{u^{n+1}_h - u^n_h}{\Delta t} \rho + \frac{\alpha}{4} ||u^{n+1}_h + u^n_h||^2 \\
\leq C' + \frac{\epsilon}{4} ||u^{n+1}_h + u^n_h||^2 + \frac{3\gamma (\Delta t)^2}{4} \left( \frac{u^{n+1}_h - u^n_h}{\Delta t} \rho \right)^2 + 3\gamma |u_{0,h}|^2
\]

\[
+ \sum_{k=0}^{n-1} \Delta t \left( \frac{1}{8} ||u^{k+1}_h + u^k_h||^2 + 3\gamma T \left( \frac{u^{k+1}_h - u^k_h}{\Delta t} \rho \right)^2 \right).
\]

Thus, for a fixed \( \epsilon < \alpha \) and any \( \Delta t \) such that \( 3\gamma (\Delta t)^2 < 4 \), we can apply Gronwall’s discrete Lemma (see, for instance, [4]) to conclude the proof. \( \square \)

**Remark 1** As a consequence of (3.2) with \( g \equiv 0 \), we obtain that the scheme is completely conservative, whereas, from (3.3), we obtain a discrete energy equality.

**4 Matrix description**

In this section we give a matrix description of algorithm 1. In particular we show a convenient way of imposing the kinematic constraint (3.1) on the fluid-solid interface.

Let \( \ell_1, \ldots, \ell_{N_F} \) be all the edges in the fluid mesh (\( N_F \) being the total number of such edges) and let \( \mathbf{\nu}_1, \ldots, \mathbf{\nu}_{N_F} \) be corresponding normal vectors. Let \( \{\phi^F_j\}_{j=1}^{N_F} \) be the Raviart-Thomas basis functions associated with each of these edges. Namely, \( \phi^F_j \) is the unique Raviart-Thomas field satisfying

\[
\frac{1}{\text{length}(\ell_i)} \int_{\ell_i} \phi^F_j \cdot \mathbf{\nu}_i = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}
\]

Let \( \{\phi^S_j\}_{j=1}^{2N_s} \) be the standard Lagrangian basis functions associated with the vertices of the solid mesh not lying on \( \Gamma_D \) (\( N_s \) being the number of such
vertices). Any $\mathbf{u}_h = (\mathbf{u}_h^s, \mathbf{u}_h^p) \in V_h$ can be written in terms of these basis:

$$
\mathbf{u}_h^s = \sum_{j=1}^{2N_S} \alpha_j \phi_j^s \quad \text{and} \quad \mathbf{u}_h^p = \sum_{j=1}^{N_F} \beta_j \phi_j^p.
$$

However, not all the pairs of this type belong to $V_h$, since they must also satisfy the kinematic constraint (2.3).

Let $K^p \overset{\text{def}}{=} \left(K^p_{ij}\right)$ and $M^p \overset{\text{def}}{=} \left(M^p_{ij}\right)$, with

$$
K^p_{ij} \overset{\text{def}}{=} \int_{\Omega_F} \rho_F c^2 \nabla \phi_j^p \cdot \nabla \phi_i^p \quad \text{and} \quad M^p_{ij} \overset{\text{def}}{=} \int_{\Omega_F} \rho_p \phi_j^p \cdot \phi_i^p,
$$

be the stiffness and mass matrices of the fluid, respectively. Let $K^s \overset{\text{def}}{=} \left(K^s_{ij}\right)$ and $M^s \overset{\text{def}}{=} \left(M^s_{ij}\right)$, with

$$
K^s_{ij} \overset{\text{def}}{=} \int_{\Omega_S} \sigma(\phi_i^s) : \varepsilon(\phi_j^s) \quad \text{and} \quad M^s_{ij} \overset{\text{def}}{=} \int_{\Omega_S} \rho_s \phi_i^s \cdot \phi_j^s,
$$

be the corresponding matrices for the solid. Finally, let $\mathbf{F}(t) \overset{\text{def}}{=} \left(F_i(t)\right)$, with

$$
F_i(t) \overset{\text{def}}{=} \int_{r_N} \mathbf{G}(t) \cdot \phi_i^S \, d\Gamma,
$$

be the time-dependent load vector.

### 4.1 Kinematic constraint

Let us assume for simplicity that the first $N_i$ basis functions in the fluid correspond to the edges lying on the interface ($N_i$ being the number of such edges). Each of the nodal components $\beta_1, \ldots, \beta_{N_i}$ can be statically condensed in terms of the nodal values of the solid displacements. In fact, for each edge of the fluid mesh $\ell_i$ lying on the interface, there holds

$$
\beta_i \overset{\text{def}}{=} \int_{\ell_i} \mathbf{u}_h^p \cdot \mathbf{v} \, d\Gamma = \int_{\ell_i} \mathbf{u}_h^s \cdot \mathbf{v} \, d\Gamma = \sum_{j=1}^{2N_S} \alpha_j \int_{\ell_i} \phi_j^S \cdot \mathbf{v} \, d\Gamma.
$$

Then, if we write $\mathbf{\beta} \overset{\text{def}}{=} \left(\beta_1, \ldots, \beta_{N_i}\right)$, $\mathbf{\beta} \overset{\text{def}}{=} \left(\beta_{N_i+1}, \ldots, \beta_{N_F}\right)$ and $\mathbf{\alpha} \overset{\text{def}}{=} \left(\alpha_1, \ldots, \alpha_{2N_S}\right)$, we have

$$
\mathbf{\beta} = \mathbf{E}\mathbf{\alpha},
$$

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where the entries of the matrix $E \defeq (E_{ij})$ are defined by

$$E_{ij} \defeq \frac{1}{\text{length}(\ell_i)} \int_{\ell_i} \phi_j^s \cdot \nu \, d\Gamma.$$ 

Hence, $\beta_i$ can be eliminated by writing

$$
\begin{pmatrix}
\alpha \\
\beta_i \\
\hat{\beta}
\end{pmatrix} = 
\begin{pmatrix}
I & 0 \\
E & 0 \\
0 & I
\end{pmatrix} 
\begin{pmatrix}
\alpha \\
\hat{\beta}
\end{pmatrix}.
$$

We split $K^\rho$ and $M^\rho$ into blocks corresponding to the unknowns $\beta_i$ and $\hat{\beta}$,

$$K^\rho = 
\begin{pmatrix}
K_i^\rho & C_K \\
C_K^t & \hat{K}^\rho
\end{pmatrix} \quad \text{and} \quad M^\rho = 
\begin{pmatrix}
M_i^\rho & C_M \\
C_M^t & \hat{M}^\rho
\end{pmatrix},$$

and we denote

$$K \defeq 
\begin{pmatrix}
K^s + E^t K^\rho E & E^t C_K \\
C_K^t E & \hat{K}^\rho
\end{pmatrix}, \quad M \defeq 
\begin{pmatrix}
M^s + E^t M^\rho E & E^t C_M \\
C_M^t E & \hat{M}^\rho
\end{pmatrix},$$

$$F^n \defeq F(t_n), \quad U_i \defeq 
\begin{pmatrix}
\alpha_i \\
\hat{\beta}_i
\end{pmatrix} \quad \text{and} \quad U^n \defeq 
\begin{pmatrix}
\alpha^n \\
\hat{\beta}^n
\end{pmatrix},$$

where $\alpha_i, \hat{\beta}_i$ are the nodal components of $u_{i,h}, i = 0, 1$, and $\alpha^n, \hat{\beta}^n$ are those of $u_h^n, n = 0, \ldots, N$. Then, the algorithm 1 can be written in terms of the genuine $N_T \defeq N_f - N_i + 2N_s$ degrees of freedom as follows:

**Algorithm 2** For $U_0, U_1 \in \mathbb{R}^{N_T}$ and $F^n \in \mathbb{R}^{N_T}, n = 0, \ldots, N$, being given data, let:

- $U^0 = U_0$,
- $U^1 \in \mathbb{R}^{N_T}$ be the solution of

$$
M \left( \frac{U^1 - U^0 - \Delta t U_1}{\Delta t^2} \right) + \frac{1}{4} K \left( U^1 + U^0 \right) = \frac{1}{4} \left( F^1 + F^0 \right),
$$

and, for $n = 1, \ldots, N - 1$,
- $U^{n+1} \in \mathbb{R}^{N_T}$ be the solution of

$$
M \left( \frac{U^{n+1} - 2U^n + U^{n-1}}{\Delta t^2} \right) + \frac{1}{4} K \left( U^{n+1} + 2U^n + U^{n-1} \right) = \frac{1}{4} \left( F^{n+1} + 2F^n + F^{n-1} \right).
$$
Remark 2 To compute the new $U_{n+1}$ at each step, one has to solve a system with associated matrix $M + \Delta t^2 K$ which is symmetric and positive definite. A factorization of such matrix can be done only once and for the rest of the time steps it is only necessary to solve two triangular linear systems.

5 Numerical experiments

We report in this section numerical results obtained with a MATLAB code that we have developed implementing the above numerical method.

For our numerical experiments we have considered a closed vessel clamped by its bottom and completely filled by fluid with its dimensions shown in Figure 2.

For the 2D vessel we have used typical physical parameters of steel:

- Density: $\rho_s = 7700 \text{ kg/m}^3$,
- Young modulus: $E = 1.44 \times 10^{11} \text{ Pa}$,
- Poisson ratio: $\nu = 0.35$.

The two latter are related to the Lame coefficients by

$$\lambda_s = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu_s = \frac{E}{2(1+\nu)}.$$ 

For the fluid filling the vessel we have used the standard parameters of water:

- Density: $\rho_f = 1000 \text{ kg/m}^3$,
- Sound speed: $c = 1430 \text{ m/s}$. 

Fig. 2. Geometrical data and initial mesh.
Table 1 shows the frequencies of the first vibration modes (i.e., those corresponding to the lowest vibration frequencies) computed with different successively refined meshes. The refinement parameter $L$ denotes the number of layers of triangles in the solid (the mesh in Figure 2 corresponds to $L = 1$).

Table 1
Computed frequencies (in Hz.) of the lowest-frequency vibration modes of the coupled system.

<table>
<thead>
<tr>
<th>Mode</th>
<th>$L = 3$</th>
<th>$L = 5$</th>
<th>$L = 7$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1^S$</td>
<td>715.4109</td>
<td>643.7516</td>
<td>619.6216</td>
<td>1.55</td>
</tr>
<tr>
<td>$\omega_2^S$</td>
<td>2049.5233</td>
<td>1880.4441</td>
<td>1824.3316</td>
<td>1.58</td>
</tr>
<tr>
<td>$\omega_3^F$</td>
<td>3774.2257</td>
<td>3523.8142</td>
<td>3427.4916</td>
<td>1.25</td>
</tr>
<tr>
<td>$\omega_4^S$</td>
<td>4498.6389</td>
<td>4324.7233</td>
<td>4266.9793</td>
<td>1.58</td>
</tr>
<tr>
<td>$\omega_5^S$</td>
<td>4757.3721</td>
<td>4478.8049</td>
<td>4373.5233</td>
<td>1.29</td>
</tr>
</tbody>
</table>

Each vibration mode in this table can be seen as a perturbation of those of the corresponding uncoupled problems: namely, either the solid in vacuo or the fluid contained in a perfectly rigid cavity. According to this, we denote the corresponding vibration frequencies $\omega_i^S$ or $\omega_i^F$, respectively.

5.1 Resonance exciting force

Our first experiment consists of starting from rest and then exciting the system with a periodic force acting on one of the edges of $\Gamma_N$ (see Figure 2). We show the response of the coupled problem during 1 second.

As a first test we excite the system during 0.5 seconds with harmonic forces $F(t, x, y) = 10^{11} y \cos(\omega t)$ with frequencies $\omega$ coinciding with two eigenfrequencies ($\omega = \omega_1^S$ and $\omega = \omega_3^S$) computed for the discrete problem with the degree of refinement $L = 3$ that we have also used for this experiment (see Table 1).

We have used approximately ten steps per period for the time discretization; more precisely, 10000 time steps for $\omega = \omega_1^S$, and 40000 time steps for $\omega = \omega_3^S$.

In Figure 3 we show the total energy of the coupled system when it is excited with harmonic forces of frequencies $\omega_1^S$ (on the left) and $\omega_3^S$ (on the right).

In both cases, the energy shows the typical parabolic pattern while the force is acting. Indeed, it is well known that, when an effect of resonance occurs, the energy of the system has a “parabolic” shape of the form:

$$\text{Energy}(t) \propto \left(A \cos^2(\omega t) + B \sin^2(\omega t)\right) t^2 + \text{small periodic terms}.$$
5.2 Non resonant exciting force

As a second test we have excited the system, starting from rest during $1/20$ seconds with a harmonic force $F(t, x, y) = 10^{11}y \cos(\omega t)$, with angular frequency $\omega = 1000$ Hz, which is far from all the eigenfrequencies of the coupled problem. Again the degree of refinement used is $L = 3$.

For the time discretization we have used 500 time steps, which is approximately eight steps per period. We show in Figure 4 the potential energy of the system

$$E_{\text{pot}}^{h,n} = \left( \frac{U^{n+1} + U^n}{2} \right)^T K \left( \frac{U^{n+1} + U^n}{2} \right).$$

In this case, the potential energy shows the typical response of a harmonic oscillator to a non-resonant excitation: a fast oscillation with a slowly varying oscillating amplitude.

5.3 Rotational initial data

Our final experiment consists in computing the response of the coupled system, in absence of exterior excitation forces, when the initial condition $u_0^k$ is a rotational mode and $u_1 \equiv 0$. The goal of this experiment is to show that, when Raviart-Thomas finite elements are used, the discrete problem is not sensitive to rotational perturbations.
Fig. 4. Potential energy for a non-resonant harmonic excitation.

For this experiment we consider a degree of refinement $L = 1$. Figure 5 shows the initial condition imposed on the fluid, which is a rotational motion in the degrees of freedom of our problem.

Fig. 5. Rotational initial condition imposed in the fluid domain.

Figure 6 shows the potential energy of the system over a period of $\frac{1}{20}$ second. We have used 500 time steps for the time discretization. It can be observed from Figure 6 that the potential energy keeps its constant value zero during the whole process. Therefore, the system remains stationary.

References

Fig. 6. Potential energy for the response to a rotational initial data.


