Approximation of a structural acoustic vibration problem by hexahedral finite elements

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A finite element method to compute elastoacoustic vibration modes in 3D problems on hexahedral meshes is analyzed. It is based on displacement formulations for fluid and solid. In order to avoid spurious modes, the discretization consists of lowest-order hexahedral Raviart-Thomas elements for the former coupled with classical trilinear isoparametric hexahedral elements for the latter. The kinematic constraint is weakly imposed and the meshes on the fluid and solid domains do not need to match on the common interface. Basic interpolation results are proved for the lowest-order hexahedral Raviart-Thomas elements. These results are used to prove convergence of the coupled finite element method, nonexistence of spurious modes, and optimal-order error estimates for eigenfunctions and eigenvalues, under the assumption that the meshes on the fluid domain are asymptotically parallelepiped. Numerical results showing sufficiency and need for this hypothesis are reported.

Keywords: Structural acoustic vibrations; finite element spectral approximation; hexahedral Raviart-Thomas elements.

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1 Introduction

The need of computing fluid-solid interactions arises in many important engineering problems. A large amount of work has been devoted to this subject during the last years. A general overview can be found in the monographs by Morand & Ohayon (1995), Conca *et al.* (1995), and Ohayon & Soize (1998), where numerical methods and further references are also given.

This paper deals with one interaction of this kind: the *elastoacoustic vibration problem*. It concerns the determination of harmonic vibrations of an elastic structure interacting with a compressible fluid. In this case displacements are small and then we can assume a linear response of the structure. We neglect gravity effects and consider a homogeneous fluid for which its reference density is constant. Other usual simplifications for this kind of problems are that viscous effects are not relevant in the fluid and that velocities are small enough for convective effects to be neglected (see Zienkiewicz & Taylor, 1991).

The problem of determining the vibrations of a fluid is usually treated by choosing pressure as primary variable. However, for coupled systems, such a choice leads to non-symmetric eigenvalue problems whose computational solution involves considerable complications (Zienkiewicz & Taylor, 1991). To avoid this drawback the fluid has been described using different variables: displacements (Kiefling & Feng, 1976), displacement potential (Morand & Ohayon, 1979), velocity potential (Everstine, 1981), or combinations of some of them.

Since the solid is generally described in terms of displacements, to choose the same variable for the fluid presents one important advantage: compatibility and equilibrium through the fluid-solid interface are satisfied automatically. This approach could be in principle applied to the solution of a broad range of problems and, after discretization, it leads to sparse symmetric matrices. Nevertheless, it is well known that the displacement formulation for the fluid suffers from the presence of zero-frequency spurious circulation modes with no physical meaning (Kiefling & Feng, 1976). After discretization by standard finite elements, these modes are approximated by other ones with non-zero frequencies interspersed among those of the physical modes.

Several approaches have been proposed to circumvent this drawback. For instance, Hamdi *et al.* (1978) introduced an irrotational constraint, taken into account by a penalty term. This technique does not attain complete elimination of spurious modes, but they are pushed towards higher frequencies. Olson & Bathe (1983) showed that this penalty formulation “locks up” for non convex fluid domains and that, although reduced integration yields some improvement, it does not assure convergence in the general case. More recently, Bathe *et al.* (1995) and Wang & Bathe (1997) proposed a mixed displacement-pressure formulation, and Gastaldi (1996) analyzed a finite element discretization, proving convergence for two-dimensional fluid domains without reentrant corners.

An alternative approach was introduced by Bermúdez & Rodríguez (1994). It consists of using standard piecewise linear elements in the solid combined with lowest-order triangular or tetrahedral *Raviart-Thomas* elements for the fluid displacements. The degrees of freedom of the latter correspond to the mean fluxes of the displacements through the element faces. The method has been proved to be free of spurious modes and to attain optimal-order convergence for any kind of fluid and solid domains (Bermúdez *et al.*, 1995; Rodríguez & Solomin, 1996). Numerical experiments showing its performance were re-

ported by Bermúdez & Rodríguez (1994) for two-dimensional (2D) triangular meshes and by Bermúdez *et al.* (1999) for three-dimensional (3D) tetrahedral ones.

However, from the point of view of efficiency, it is well known that, for the same number of degrees of freedom, numerical solutions computed by means of hexahedral elements approach the exact solution of structural problems better than those computed by means of tetrahedral ones. Furthermore, using Raviart-Thomas elements in the fluid, the number of degrees of freedom (i.e., the number of faces) for hexahedra is approximately only one quarter of that for tetrahedra on meshes with the same vertices. Numerical evidence of the advantage of using hexahedral meshes in the elastoacoustic vibration problem can be found in Bermúdez *et al.* (2001a, 2001b).

The aim of this paper is to analyze the numerical method used in these last two references. This method is based on hexahedral finite elements: trilinear isoparametric in the solid and lowest-order Raviart-Thomas in the fluid. Fluid and solid meshes do not need to match on the common interface because the kinematic constraint (i.e., equal normal displacements for fluid and solid) is imposed in a weak sense. The analysis of trilinear isoparametric elements is rather standard; only a Clément's type interpolation is needed (see Clément, 1975). However this is not the case for Raviart-Thomas hexahedral elements.

Raviart-Thomas elements are $H(\text{div})$ -conforming. They provide optimal-order approximations in $H(\text{div})$ of smooth vector fields on 2D shape-regular rectangular meshes (Raviart & Thomas, 1977) or 3D cubic meshes (Nédélec, 1980). A detailed analysis of these elements in 2D has been carried out by Thomas (1977), where optimal-order L^2 approximation results were proved for general shape-regular quadrilaterals. However, similar optimal results in $H(\text{div})$ were proved in the same reference only for *asymptotically parallelogram* meshes (i.e., meshes built on quadrilaterals such that the distance of the vertices of each element to those of a parallelogram are higher order terms). To the best of the authors' knowledge, a similar analysis has not been performed on hexahedral 3D meshes.

On the other hand, it was recently noted by Arnold *et al.* (2001, 2002a) that the extension to general quadrilaterals of convergence results valid for rectangular elements is not straightforward. Moreover, the order of convergence can deteriorate when non-standard finite elements are used in distorted quadrilaterals, even though they satisfy the usual shape-regularity assumption. In particular, Arnold *et al.* (2002b) considered the approximation properties of quadrilateral finite element spaces of vector-valued functions defined by the *Piola transform*, as is the case with Raviart-Thomas elements. For these elements, they showed degradation of the convergence order for some trapezoid meshes as compared to rectangular ones.

In the present paper we prove $H(\text{div})$ approximation results for lowest-order Raviart-Thomas hexahedral elements on general shape-regular 3D meshes. The error estimates on each element depend on a parameter which measures the deviation of the hexahedron with respect to a parallelepiped and are optimal-order on *asymptotically parallelepiped* meshes (i.e., on meshes such that these parameters are higher order as compared with the diameter of the element). This is used to analyze the spectral approximation properties of the coupled finite element method applied to the elastoacoustic vibration problem on meshes which are asymptotically parallelepiped on the fluid domain. We demonstrate the need for these kinds of meshes by means of a numerical test.

The outline of the paper is as follows. The spectral problem to be solved and the main

theoretical results are recalled in Section 2. In Section 3, we introduce the finite element spaces and prove basic interpolation results. To this aim, we begin by studying the approximation properties of the lowest-order Raviart-Thomas hexahedron. In Section 4, we introduce the discrete spectral problem and prove convergence and nonexistence of spurious modes, as well as optimal-order error estimates for eigenfunctions and eigenvalues, assuming that the elements in the meshes on the fluid domain are, asymptotically, parallelepiped. Finally, numerical results showing sufficiency and need for this hypothesis are reported in Section 5.

2 Statement of the problem

We consider the problem of determining the free vibration modes of a linear elastic structure containing an acoustic (barotropic, inviscid, and compressible) fluid.

Let Ω_F and Ω_S denote polygonal or polyhedral bounded domains in $\mathbb{R}^n$ ($n = 2, 3$) occupied by the fluid and the solid, respectively, as shown in Fig. 1 in the 2D case. Let $\Gamma_I = \partial\Omega_F$ be the interface between the two media, and $\mathbf{n}$ its unit normal vector pointing outward from Ω_F . The exterior boundary of the solid domain is the union of Γ_D and Γ_N : the structure is fixed along Γ_D and free of stress along Γ_N ; we assume that the 2D-measure of Γ_D is strictly positive. Finally, $\boldsymbol{\eta}$ denotes the unit outward normal vector along Γ_N .

FIG. 1: *Fluid and solid domains.*

Throughout this paper we will use standard notation for Sobolev spaces, norms, and seminorms. We also denote $H_{\Gamma_D}^1(\Omega_S)$, the closed subspace of functions in $H^1(\Omega_S)$ with a vanishing trace on Γ_D and $H(\text{div}, \Omega_F) := \{\mathbf{u} \in L^2(\Omega_F)^n : \text{div } \mathbf{u} \in L^2(\Omega_F)\}$, endowed with its natural norm that we denote $\|\cdot\|_{\text{div}, \Omega_F}$.

We use the following notations for the physical magnitudes; in the fluid:

- $\mathbf{u}$: the displacement vector,
- p : the pressure,
- ρ_F : the density,
- c : the acoustic speed,

and in the solid:

- $\mathbf{w}$: the displacement vector,
- ρ_s : the density,
- λ_s and μ_s : the Lamé coefficients,
- $\epsilon(\mathbf{w})$: the strain tensor defined by $\epsilon_{ij}(\mathbf{w}) := \frac{1}{2} \left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} \right)$ $i, j = 1, \dots, n$,
- $\sigma(\mathbf{w})$: the stress tensor which is related to the strain tensor by Hooke's law:

$$\sigma_{ij}(\mathbf{w}) := \lambda_s \sum_{k=1}^n \epsilon_{kk}(\mathbf{w}) \delta_{ij} + 2\mu_s \epsilon_{ij}(\mathbf{w}), \quad i, j = 1, \dots, n.$$

The classical elastoacoustics model for small amplitude motions yields the following eigenvalue problem for the free vibration modes of the coupled system and their corresponding vibration frequencies ω (see, for instance, Morand & Ohayon, 1995):

Problem 2.1 Find $\omega > 0$, $\mathbf{u} \in H(\text{div}, \Omega_F)$, $\mathbf{w} \in H^1(\Omega_S)^n$, and $p \in H^1(\Omega_F)$, $(\mathbf{u}, \mathbf{w}, p) \neq (\mathbf{0}, \mathbf{0}, 0)$, such that:

$$\nabla p - \omega^2 \rho_F \mathbf{u} = \mathbf{0} \quad \text{in } \Omega_F, \quad (2.1)$$

$$p + \rho_F c^2 \text{div } \mathbf{u} = 0 \quad \text{in } \Omega_F, \quad (2.2)$$

$$\text{Div} [\sigma(\mathbf{w})] + \omega^2 \rho_s \mathbf{w} = \mathbf{0} \quad \text{in } \Omega_S, \quad (2.3)$$

$$\sigma(\mathbf{w}) \mathbf{n} + p \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_I, \quad (2.4)$$

$$\mathbf{w} \cdot \mathbf{n} - \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_I, \quad (2.5)$$

$$\sigma(\mathbf{w}) \boldsymbol{\eta} = \mathbf{0} \quad \text{on } \Gamma_N, \quad (2.6)$$

$$\mathbf{w} = \mathbf{0} \quad \text{on } \Gamma_D. \quad (2.7)$$

The coupling between the fluid and the structure is taken into account by equations (2.4) and (2.5). The first one relates the solid stress on the interface with the pressure exerted by the fluid. The second one is a slipping condition which implies that fluid and solid are in contact at the interface. Both equations can be understood in the L^2 sense since $p|_{\Gamma_I}$ and $\mathbf{w}|_{\Gamma_I} \cdot \mathbf{n}$, both belong to $L^2(\Gamma_I)$.

We introduce appropriate function spaces to obtain a variational formulation for this problem. Let

$$\mathcal{H} := L^2(\Omega_F)^n \times L^2(\Omega_S)^n \quad \text{and} \quad \mathcal{X} := H(\text{div}, \Omega_F) \times H_{\Gamma_D}^1(\Omega_S)^n,$$

both endowed with their respective product norms which we denote by $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{X}}$, respectively. Let

$$\mathcal{V} := \{(\mathbf{u}, \mathbf{w}) \in \mathcal{X} : \mathbf{u} \cdot \mathbf{n} = \mathbf{w} \cdot \mathbf{n} \quad \text{on } \Gamma_I\}.$$

This closed subspace of $\mathcal{X}$ is the set of *kinematically admissible* coupled fluid-solid displacements.

Let a and b be the symmetric continuous bilinear forms defined by

$$a((\mathbf{u}, \mathbf{w}), (\mathbf{v}, \mathbf{z})) := \int_{\Omega_F} \rho_F c^2 \text{div } \mathbf{u} \text{div } \mathbf{v} + \int_{\Omega_S} \sigma(\mathbf{w}) : \epsilon(\mathbf{z}), \quad (\mathbf{u}, \mathbf{w}), (\mathbf{v}, \mathbf{z}) \in \mathcal{X},$$

and

$$b((\mathbf{u}, \mathbf{w}), (\mathbf{v}, \mathbf{z})) := \int_{\Omega_F} \rho_F \mathbf{u} \cdot \mathbf{v} + \int_{\Omega_S} \rho_s \mathbf{w} \cdot \mathbf{z}, \quad (\mathbf{u}, \mathbf{w}), (\mathbf{v}, \mathbf{z}) \in \mathcal{H}.$$

If we integrate by parts (2.1) and (2.3) multiplied by $\mathbf{v} \in H(\text{div}, \Omega_F)$ and $\mathbf{z} \in H_{\Gamma_D}^1(\Omega_S)^n$, respectively, eliminate the pressure p by means of (2.2) and (2.4), use (2.6), and denote $\lambda = \omega^2$, then we obtain the following symmetric variational spectral problem:

Problem 2.2 Find $\lambda \in \mathbb{R}$ and $(\mathbf{u}, \mathbf{w}) \in \mathcal{V}$, $(\mathbf{u}, \mathbf{w}) \neq (\mathbf{0}, \mathbf{0})$, such that

$$a((\mathbf{u}, \mathbf{w}), (\mathbf{v}, \mathbf{z})) = \lambda b((\mathbf{u}, \mathbf{w}), (\mathbf{v}, \mathbf{z})) \quad \forall (\mathbf{v}, \mathbf{z}) \in \mathcal{V}.$$

It is easy to show that $\lambda_0 = 0$ is an eigenvalue of this problem with corresponding eigenspace

$$\mathcal{K} := \{(\mathbf{u}, \mathbf{0}) \in \mathcal{V} : \text{div } \mathbf{u} = 0 \text{ in } \Omega_F\}.$$

This infinite-dimensional eigenspace consists of pure rotational motions which induce neither fluid pressure variations nor structural vibrations. These eigenfunctions arise because the irrotational constraint implicit in (2.1) for $\omega \neq 0$ has not been imposed on the fluid displacements allowed in the model.

The remaining vibration frequencies are strictly positive and correspond to actual vibrations of the coupled fluid-solid system. The whole spectrum of the problem can be characterized by using the spectral theorem for compact operators and the fact that the fluid displacements associated with $\lambda_0 = 0$ are orthogonal to any irrotational fluid displacement of the form $\mathbf{u} = \nabla \varphi$. More precisely,

$$\mathcal{G} := \{(\mathbf{u}, \mathbf{w}) \in \mathcal{V} : \mathbf{u} = \nabla \varphi, \varphi \in H^1(\Omega_F)\}$$

is the orthogonal complement of $\mathcal{K}$ in $\mathcal{V}$ with respect to the inner product b . Notice that $\mathcal{K}$ and $\mathcal{G}$ are orthogonal in the Hilbert products of $\mathcal{H}$ and $\mathcal{X}$, too.

Given $(\mathbf{u}, \mathbf{w}) \in \mathcal{G}$, let $\varphi \in H^1(\Omega_F)$ be such that $\mathbf{u} = \nabla \varphi$; this function φ is the fluid displacement potential and it satisfies the following compatible Neumann problem:

$$\begin{cases} \Delta \varphi = \text{div } \mathbf{u} & \text{in } \Omega_F, \\ \frac{\partial \varphi}{\partial \mathbf{n}} = \mathbf{w} \cdot \mathbf{n} & \text{on } \Gamma_1. \end{cases} \quad (2.8)$$

This problem attains a unique solution $\varphi \in H^1(\Omega_F)/\mathbb{R}$ and

$$\|\mathbf{u}\|_{0, \Omega_F} = \|\nabla \varphi\|_{0, \Omega_F} \leq C \left(\|\text{div } \mathbf{u}\|_{0, \Omega_F} + \|\mathbf{w}\|_{1, \Omega_S} \right),$$

where C is a constant independent of $\mathbf{u}$ and $\mathbf{w}$. Henceforth, C denotes a generic strictly positive constant which is not necessarily the same at each occurrence.

As an immediate consequence of this estimate and Korn's inequality, the bilinear form a is $\mathcal{X}$ -elliptic on $\mathcal{G}$. This allows us to define the linear operator $T : \mathcal{H} \rightarrow \mathcal{G}$ as follows: Given $(\mathbf{f}, \mathbf{g}) \in \mathcal{H}$, let $T(\mathbf{f}, \mathbf{g}) := (\mathbf{u}, \mathbf{w})$, with $(\mathbf{u}, \mathbf{w}) \in \mathcal{G}$ being the solution of

$$a((\mathbf{u}, \mathbf{w}), (\mathbf{v}, \mathbf{z})) = b((\mathbf{f}, \mathbf{g}), (\mathbf{v}, \mathbf{z})) \quad \forall (\mathbf{v}, \mathbf{z}) \in \mathcal{G}.$$

The linear operator T is self-adjoint with respect to a and b . Moreover, $T|_{\mathcal{G}}$ is compact in $\|\cdot\|_{\mathcal{H}}$. Then, from Riesz-Fredholm theory, we know that the spectrum of $T|_{\mathcal{G}}$ consists of $\mu = 0$ and a sequence of finite-multiplicity eigenvalues $\mu_n > 0$, $n \in \mathbb{N}$, converging to

0. Moreover, λ is a positive eigenvalue of Problem 2.2 if and only if $\mu = 1/\lambda$ is a positive eigenvalue of the operator T , and the corresponding associated eigenfunctions coincide.

The fluid displacement $\mathbf{u}$ corresponding to any pair $(\mathbf{u}, \mathbf{w}) \in \mathcal{G}$ satisfies additional regularity. Indeed, because of the standard *a priori* estimate for the Neumann problem (2.8) (see Dauge, 1988), there exist constants $r \in (\frac{1}{2}, 1]$ and $C > 0$, independent of $\mathbf{u}$ and $\mathbf{w}$, such that $\mathbf{u} = \nabla \varphi \in H^r(\Omega_F)^n$ and

$$\|\mathbf{u}\|_{r, \Omega_F} \leq C \left(\|\operatorname{div} \mathbf{u}\|_{0, \Omega_F} + \|\mathbf{w}\|_{1, \Omega_S} \right). \quad (2.9)$$

Remark 2.3 *The regularity constant r of the Neumann problem (2.8) only depends on the domain Ω_F . If Ω_F is convex, then $r = 1$; otherwise, $\frac{1}{2} < r < 1$. In the last case, in 2D, (2.9) holds $\forall r < \frac{\pi}{\theta}$, with θ being the largest reentrant corner of Ω_F .*

The following theorem, which shows that T is a regularizing operator, was proved by Bermúdez *et al.* (1995) in 2D. A similar proof based on inequality (2.9) and the *a priori* estimate for the linear elasticity problem (see Dauge, 1988) is valid in 3D, too.

Theorem 2.4 *There exist constants $r \in (\frac{1}{2}, 1]$, $s \in (0, 1]$, and $C > 0$, such that $\forall (\mathbf{f}, \mathbf{g}) \in \mathcal{H}$, if $(\mathbf{u}, \mathbf{w}) = T(\mathbf{f}, \mathbf{g})$, then $\mathbf{u} \in H^r(\Omega_F)^n$, $\operatorname{div} \mathbf{u} \in H^1(\Omega_F)$, $\mathbf{w} \in H^{1+s}(\Omega_S)^n$, and*

$$\|\mathbf{u}\|_{r, \Omega_F} + \|\operatorname{div} \mathbf{u}\|_{1, \Omega_F} + \|\mathbf{w}\|_{1+s, \Omega_S} \leq C \|(\mathbf{f}, \mathbf{g})\|_{\mathcal{H}}.$$

3 Finite element spaces

In this section we introduce finite element spaces to discretize the fluid and solid displacement fields appearing in Problem 2.2 and discuss some of their approximation properties. Although most of what follows is valid for both, $n = 2$ and $n = 3$, for definiteness we present our results in a 3D setting and include some explicit comments on those aspects that differ in the 2D case.

We consider partitions in hexahedra of Ω_F and Ω_S and standard Lagrangian trilinear isoparametric hexahedral elements to discretize the solid displacement field. However, it is well known that using Lagrangian elements for the fluid displacements produce spurious modes: rotational modes with non-zero frequencies interspersed among those of the irrotational ones (Kiefling & Feng, 1976; Hamdi *et al.*, 1978). To avoid this drawback, we use lowest-order *Raviart-Thomas* hexahedral finite elements instead. These elements have been introduced by Raviart & Thomas (1977) in 2D meshes built on triangles or quadrilaterals and extended by Nédélec (1980) to 3D meshes built on tetrahedra or cubes.

In what follows we recall the definition and prove some approximation properties of lowest-order *Raviart-Thomas* hexahedral finite elements. Then we introduce the discrete space to numerically solve Problem 2.2 and obtain error estimates for an interpolant on this space.

3.1 Hexahedra

A *hexahedron* is the image of the unit cube $\widehat{K} := [0, 1]^3$ under an invertible trilinear map defined on $\widehat{K}$. (Throughout the paper, we use “hatted” variables to denote magnitudes in

the reference element $\widehat{K}$.) Indeed, let $\mathbf{F}_K : \widehat{K} \rightarrow \mathbb{R}^3$ be a trilinear one-to-one map with a strictly positive *Jacobian*:

$$J_K(\widehat{\mathbf{x}}) := \det [\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}})] > 0 \quad \forall \widehat{\mathbf{x}} \in \widehat{K},$$

where $\widehat{D}\mathbf{F}_K$ denotes the *Jacobian matrix* of the mapping $\mathbf{F}_K$. Then the corresponding hexahedron is $K := \mathbf{F}_K(\widehat{K})$ (see Fig. 2). Notice that K does not need to have planar faces; the edges are always straight segments, but each face is planar if and only if its four vertices lie on a plane.

FIG. 2: Trilinear isoparametric mapping.

In practice, the possibility of having curved faces allows a better fitting of curved boundaries by judiciously choosing the vertices of the corresponding faces with the only restriction of lying on the boundary. However, this is beyond the scope of the present theoretical analysis, because we have assumed that the domains are polyhedral.

Let h_K be the *diameter* of K . Let $\bar{\sigma}_K$ and $\underline{\sigma}_K$ be strictly positive constants such that

$$\frac{|\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}})|}{h_K} \leq \bar{\sigma}_K \quad \forall \widehat{\mathbf{x}} \in \widehat{K} \quad \text{and} \quad \frac{|D(\mathbf{F}_K^{-1})(\mathbf{x})|^{-1}}{h_K} \geq \underline{\sigma}_K \quad \forall \mathbf{x} \in K. \quad (3.1)$$

Here and thereafter, $|\cdot|$ denotes the standard *Euclidean norm* in $\mathbb{R}^n$ as well as the corresponding subordinate matrix norm in $\mathbb{R}^{n \times n}$ ($n = 2, 3$). Notice that the constants $\bar{\sigma}_K$ and $\underline{\sigma}_K$ measure the shape-regularity of the hexahedron.

Since $[\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}})]^{-1} = D(\mathbf{F}_K^{-1})(\mathbf{F}_K(\widehat{\mathbf{x}}))$, the above definitions yield the estimates:

$$\begin{aligned} \underline{\sigma}_K h_K &\leq |\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}})| \leq \bar{\sigma}_K h_K & \forall \widehat{\mathbf{x}} \in \widehat{K}, \\ \frac{1}{\bar{\sigma}_K h_K} &\leq |D(\mathbf{F}_K^{-1})(\mathbf{x})| \leq \frac{1}{\underline{\sigma}_K h_K} & \forall \mathbf{x} \in K. \end{aligned}$$

On the other hand, $|D(\mathbf{F}_K^{-1})(\mathbf{F}_K(\widehat{\mathbf{x}}))|^{-1} \leq [J_K(\widehat{\mathbf{x}})]^{\frac{1}{3}} \leq |\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}})|$ and, consequently,

$$\underline{\sigma}_K^3 h_K^3 \leq J_K(\widehat{\mathbf{x}}) \leq \bar{\sigma}_K^3 h_K^3 \quad \forall \widehat{\mathbf{x}} \in \widehat{K}. \quad (3.2)$$

The *cofactor matrix* of $\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}})$ satisfies $\text{Cof}[\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}})]^t = J_K(\widehat{\mathbf{x}})[\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}})]^{-1}$. Hence, the estimates above lead to the following bound:

$$\frac{\sigma_K^3}{\bar{\sigma}_K} h_K^2 \leq |\text{Cof}(\widehat{D}\mathbf{F}_K)(\widehat{\mathbf{x}})| \leq \frac{\bar{\sigma}_K^3}{\sigma_K} h_K^2 \quad \forall \widehat{\mathbf{x}} \in \widehat{K}. \quad (3.3)$$

To measure the “deviation” of a hexahedron K with respect to a parallelepiped, we decompose $\mathbf{F}_K$ as a sum of two parts, one affine and the other depending on four geometrical vector parameters, $\varepsilon_1, \dots, \varepsilon_4 \in \mathbb{R}^3$ (see Fig. 3). Indeed, let $\alpha_i := \mathbf{F}_K(\mathbf{e}_i) \in \mathbb{R}^3$, with $\mathbf{e}_i$ being the i^{th} canonical vector, $i = 1, 2, 3$. For $\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$, let $\mathbf{F}_K(\widehat{\mathbf{x}}) = \mathbf{F}_K^l(\widehat{\mathbf{x}}) + \mathbf{F}_K^d(\widehat{\mathbf{x}})$, with

$$\mathbf{F}_K^l(\widehat{\mathbf{x}}) := \widehat{x}_1 \alpha_1 + \widehat{x}_2 \alpha_2 + \widehat{x}_3 \alpha_3, \quad (3.4)$$

$$\mathbf{F}_K^d(\widehat{\mathbf{x}}) := \widehat{x}_2 \widehat{x}_3 \varepsilon_1 + \widehat{x}_1 \widehat{x}_3 \varepsilon_2 + \widehat{x}_1 \widehat{x}_2 \varepsilon_3 + \widehat{x}_1 \widehat{x}_2 \widehat{x}_3 (\varepsilon_4 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \quad (3.5)$$

where $\mathbf{F}_K^l$ is the linear part and $\mathbf{F}_K^d$ the deviation (for simplicity, we have not considered translations). The geometrical meaning of the vector parameters $\varepsilon_1, \dots, \varepsilon_4$ can be seen from Fig. 3. Let us recall that if the faces of the element K are planar, then $\varepsilon_4 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ and the cubic term in (3.5) vanishes.

FIG. 3: Image of $\widehat{K}$ by $\mathbf{F}_K^l$ (left) and $\mathbf{F}_K$ decomposed as $\mathbf{F}_K^l + \mathbf{F}_K^d$ (right).

The following parameter measures the deviation of K with respect to a parallelepiped:

$$\varepsilon_K := \max_{1 \leq i \leq 4} |\varepsilon_i|. \quad (3.6)$$

The derivatives of J_K and $\text{Cof}[\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}})]$ are bounded in terms of ε_K and h_K as follows:

Lemma 3.1 *There exists a strictly positive constant C such that the following estimates hold true for any hexahedron $K \subset \mathbb{R}^3$:*

$$\max_{\widehat{\mathbf{x}} \in \widehat{K}} |\widehat{\nabla} J_K(\widehat{\mathbf{x}})| \leq C \varepsilon_K h_K^2, \quad (3.7)$$

$$\max_{\widehat{\mathbf{x}} \in \widehat{K}} \left| \frac{\partial}{\partial \widehat{x}_k} \left(\text{Cof}[\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}})] \right)_{ij} \right| \leq C \varepsilon_K h_K, \quad 1 \leq i, j, k \leq 3. \quad (3.8)$$

Proof. Given a hexahedron K , let $\alpha_i \in \mathbb{R}^3$, $i = 1, 2, 3$ and $\varepsilon_j \in \mathbb{R}^3$, $j = 1, \dots, 4$, be as above. Explicit computations lead to

$$\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}}) = (\alpha_1 | \alpha_2 | \alpha_3) + \mathbf{P}(\widehat{\mathbf{x}}),$$

where $(\alpha_1 | \alpha_2 | \alpha_3)$ denotes the matrix whose columns are the vectors α_i , $i = 1, 2, 3$, and, using the same notation,

$$\mathbf{P}(\widehat{\mathbf{x}}) := (\widehat{x}_3\varepsilon_2 + \widehat{x}_2\varepsilon_3 + \widehat{x}_2\widehat{x}_3\varepsilon_0 | \widehat{x}_3\varepsilon_1 + \widehat{x}_1\varepsilon_3 + \widehat{x}_1\widehat{x}_3\varepsilon_0 | \widehat{x}_2\varepsilon_1 + \widehat{x}_1\varepsilon_2 + \widehat{x}_1\widehat{x}_2\varepsilon_0),$$

with $\varepsilon_0 := \varepsilon_4 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$. Therefore,

$$J_K(\widehat{\mathbf{x}}) = \det(\alpha_1 | \alpha_2 | \alpha_3) + p(\widehat{\mathbf{x}}),$$

with p being a polynomial function of $\widehat{\mathbf{x}}$. All the coefficients of p are in turn homogeneous cubic polynomials in $\alpha_1, \alpha_2, \alpha_3, \varepsilon_1, \dots, \varepsilon_4$. Each monomial of these cubic polynomials is the product of three entries taken from $\alpha_1, \alpha_2, \alpha_3$ or $\varepsilon_1, \dots, \varepsilon_4$, with at least one of them being from $\varepsilon_1, \dots, \varepsilon_4$. Clearly, $|\alpha_i| \leq h_K$, $i = 1, 2, 3$. Moreover, $|\varepsilon_i| \leq 3h_K$, $i = 1, 2, 3$, and $|\varepsilon_4| \leq 4h_K$. Indeed, for instance $|\varepsilon_3| \leq 3h_K$, because $|\alpha_1 + \alpha_2 + \varepsilon_3| \leq h_K$ (see Fig. 3) and $|\alpha_1 + \alpha_2| \leq 2h_K$. Hence, since $\forall \widehat{\mathbf{x}} \in \widehat{K}$, $0 \leq \widehat{x}_i \leq 1$, $i = 1, 2, 3$, we conclude that $\exists C > 0$ such that (3.7) holds true.

Analogously,

$$\text{Cof}[\widehat{D}\mathbf{F}_K(\widehat{\mathbf{x}})] = \text{Cof}(\alpha_1 | \alpha_2 | \alpha_3) + \widetilde{\mathbf{P}}(\widehat{\mathbf{x}}),$$

where the entries of $\widetilde{\mathbf{P}}(\widehat{\mathbf{x}}) = (\widetilde{p}_{ij}(\widehat{\mathbf{x}}))$ are polynomial functions of $\widehat{\mathbf{x}}$. All the coefficients of each $\widetilde{p}_{ij}$, $i, j = 1, 2, 3$, are in turn homogeneous quadratic polynomials in $\alpha_1, \alpha_2, \alpha_3, \varepsilon_1, \dots, \varepsilon_4$. Each monomial of these quadratic polynomials is the product of two entries taken from $\alpha_1, \alpha_2, \alpha_3$ or $\varepsilon_1, \dots, \varepsilon_4$, with at least one of them being from $\varepsilon_1, \dots, \varepsilon_4$. Hence, the same arguments as above lead us to conclude that $\exists C > 0$ such that (3.8) holds true, too. Thus we conclude the proof. $\square$

Remark 3.2 *Similar estimates were proved by Thomas (1977) in 2D. In this case there is a unique vector parameter $\varepsilon \in \mathbb{R}^2$ which measures the deviation of a quadrilateral K with respect to a parallelogram. Actually, Thomas (1977) used another one: the vector δ joining the midpoints of the diagonals of K . However, it is simple to show that $\delta = \frac{1}{2}\varepsilon$.*

3.2 The lowest-order Raviart-Thomas hexahedron

Given a vector field $\mathbf{u}$ defined on a hexahedron K , Raviart-Thomas elements discretize the whole vector field instead of its components separately. The degrees of freedom of the lowest-order Raviart-Thomas finite elements are the mean values of the fluxes of $\mathbf{u}$ on each face of the hexahedron. To define these elements, we change coordinates to the reference unit cube and use the *Piola transform* associated with this change of coordinates.

Let $\mathbf{F}_K$ be the trilinear one-to-one map applying $\widehat{K}$ onto K as introduced above (see Fig. 2). We recall that the Piola transform associated with $\mathbf{F}_K$ allows transforming a vector

field from the current element to the reference one, leaving invariant its flux through any surface. Given a vector field $\mathbf{u}$ defined on K , its Piola transform is defined by

$$\hat{\mathbf{u}}(\hat{\mathbf{x}}) := J_K(\hat{\mathbf{x}}) [\hat{D}\mathbf{F}_K(\hat{\mathbf{x}})]^{-1} \mathbf{u}(\mathbf{F}_K(\hat{\mathbf{x}})), \quad \hat{\mathbf{x}} \in \hat{K}.$$

Notice that the Piola transform can be equivalently written as $\hat{\mathbf{u}} = \text{Cof}(\hat{D}\mathbf{F}_K)^t (\mathbf{u} \circ \mathbf{F}_K)$. Its main property is that, for any surface $\hat{S} \subset \hat{K}$, if $S = \mathbf{F}_K(\hat{S})$, then

$$\int_{\hat{S}} \hat{\mathbf{u}} \cdot \mathbf{n}_{\hat{S}} = \int_S \mathbf{u} \cdot \mathbf{n}_S, \quad (3.9)$$

where, from now on, $\mathbf{n}_S$ (as well as $\mathbf{n}_{\hat{S}}$) denotes a unit vector normal to the surface S (resp. $\hat{S}$). Hence, via the Gauss theorem, for any volume $\hat{V} \subset \hat{K}$, if $V = \mathbf{F}_K(\hat{V})$, then $\int_{\hat{V}} \widehat{\text{div}} \hat{\mathbf{u}} = \int_V \text{div} \mathbf{u}$ and, consequently,

$$\widehat{\text{div}} \hat{\mathbf{u}} = J_K (\text{div} \mathbf{u}) \circ \mathbf{F}_K \quad \text{in } \hat{K}. \quad (3.10)$$

A lowest-order Raviart-Thomas element on K is a vector field $\mathbf{u}$ whose Piola transform $\hat{\mathbf{u}}$ satisfies

$$\hat{\mathbf{u}} \in \hat{\mathcal{R}} := \mathcal{P}_{1,0,0}(\hat{K}) \times \mathcal{P}_{0,1,0}(\hat{K}) \times \mathcal{P}_{0,0,1}(\hat{K}),$$

where $\mathcal{P}_{i,j,k}(\hat{K})$ denotes the set of polynomial functions defined on $\hat{K}$ of degrees i, j , and k in $\hat{x}_1, \hat{x}_2$, and $\hat{x}_3$, respectively. Thus, the space of lowest-order Raviart-Thomas finite elements on K is

$$\mathcal{R}_K := \left\{ \mathbf{u} : K \rightarrow \mathbb{R}^3 : \hat{\mathbf{u}} \in \hat{\mathcal{R}} \right\}.$$

The vector fields in $\hat{\mathcal{R}}$ have constant normal components on each face of $\hat{K}$. Moreover, these constant normal components determine a unique vector field of this type. Thus, by virtue of (3.9), the mean values of the fluxes of $\mathbf{u}$ on each face of K can be taken as the degrees of freedom defining this element (see Fig. 4).

FIG. 4: Degrees of freedom of the lowest-order hexahedral Raviart-Thomas elements.

Given $\mathbf{u} \in H^r(K)^3$, $r > \frac{1}{2}$, the Raviart-Thomas interpolant of $\mathbf{u}$ in K is defined as the unique $\mathbf{u}_K \in \mathcal{R}_K$ satisfying

$$\int_F \mathbf{u}_K \cdot \mathbf{n}_F = \int_F \mathbf{u} \cdot \mathbf{n}_F \quad (3.11)$$

for all faces F of K . The mapping $\mathbf{u} \mapsto \mathbf{u}_K$ extends by duality arguments to $\mathbf{u} \in \mathbf{H}(\operatorname{div}, K) \cap \mathbf{H}^r(K)^3 \ \forall r > 0$, defining a bounded linear operator (see Brezzi & Fortin, 1991).

Error estimates for this interpolation in 2D were proved by Thomas (1977) for $\mathbf{u} \in \mathbf{H}^1(K)^3$ with $\operatorname{div} \mathbf{u} \in \mathbf{H}^1(K)$. In what follows we adapt the arguments in this reference to 3D and fractional Sobolev spaces to prove the following result:

Lemma 3.3 *If $\mathbf{u} \in \mathbf{H}(\operatorname{div}, K) \cap \mathbf{H}^r(K)^3$ with $0 < r \leq 1$, then there exists a strictly positive constant C , only depending on $\bar{\sigma}_K$, $\underline{\sigma}_K$, and r , such that*

$$\|\mathbf{u} - \mathbf{u}_K\|_{0,K} \leq C \left(\frac{\varepsilon_K}{h_K} \|\mathbf{u}\|_{0,K} + h_K^r |\mathbf{u}|_{r,K} + h_K \|\operatorname{div} \mathbf{u}\|_{0,K} \right).$$

If moreover $\operatorname{div} \mathbf{u} \in \mathbf{H}^q(K)$ with $0 < q \leq 1$, then there exists a strictly positive constant C , only depending on $\bar{\sigma}_K$, $\underline{\sigma}_K$, and q , such that

$$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_K)\|_{0,K} \leq C \left(\frac{\varepsilon_K}{h_K} \|\operatorname{div} \mathbf{u}\|_{0,K} + h_K^q |\operatorname{div} \mathbf{u}|_{q,K} \right).$$

Proof. For $\mathbf{u} \in \mathbf{H}^r(K)^3$ with $0 < r \leq 1$, let $\widehat{\mathbf{u}}$ be its Piola transform. Since $\operatorname{Cof}(\widehat{D}\mathbf{F}_K)$ is smooth in $\widehat{K}$, straightforward computations allow us to show that

$$|\widehat{\mathbf{u}}|_{r,\widehat{K}} \leq |\operatorname{Cof}(\widehat{D}\mathbf{F}_K)|_{1,\infty,\widehat{K}} \|\mathbf{u} \circ \mathbf{F}_K\|_{0,\widehat{K}} + \|\operatorname{Cof}(\widehat{D}\mathbf{F}_K)\|_{0,\infty,\widehat{K}} |\mathbf{u} \circ \mathbf{F}_K|_{r,\widehat{K}},$$

where, for $r < 1$, we have used the intrinsic norm of $\mathbf{H}^r(\widehat{K})$ (see, for instance, Girault & Raviart, 1986). Explicit calculations yield the following estimates:

$$\begin{aligned} \|\mathbf{u} \circ \mathbf{F}_K\|_{0,\widehat{K}} &\leq \frac{1}{(\min_{\widehat{\mathbf{x}} \in \widehat{K}} J_K)^{\frac{1}{2}}} \|\mathbf{u}\|_{0,K}, \\ |\mathbf{u} \circ \mathbf{F}_K|_{r,\widehat{K}} &\leq \frac{\left(\max_{\widehat{\mathbf{x}} \in \widehat{K}} |\widehat{D}\mathbf{F}_K|\right)^{\frac{3}{2}+r}}{\min_{\widehat{\mathbf{x}} \in \widehat{K}} J_K} |\mathbf{u}|_{r,K}, \end{aligned}$$

where the intrinsic norms of non-integer Sobolev spaces have been used again for the latter, when $r < 1$. Consequently, $\widehat{\mathbf{u}} \in \mathbf{H}^r(\widehat{K})^3$ and (3.1), (3.2), (3.3), and (3.8) lead to

$$|\widehat{\mathbf{u}}|_{r,\widehat{K}} \leq C h_K^{\frac{1}{2}} \left(\frac{\varepsilon_K}{h_K} \|\mathbf{u}\|_{0,K} + h_K^r |\mathbf{u}|_{r,K} \right),$$

with C only depending on $\bar{\sigma}_K$, $\underline{\sigma}_K$, and r .

On the other hand, for $\mathbf{u} \in \mathbf{H}(\operatorname{div}, K)$, (3.10) and (3.2) imply

$$\|\widehat{\operatorname{div} \mathbf{u}}\|_{0,\widehat{K}} \leq \bar{\sigma}_K^{\frac{3}{2}} h_K^{\frac{3}{2}} \|\operatorname{div} \mathbf{u}\|_{0,K}$$

and then $\widehat{\mathbf{u}} \in \mathbf{H}(\operatorname{div}, \widehat{K})$. Hence the Raviart-Thomas interpolant $\widehat{\mathbf{u}}_{\widehat{K}}$ is well defined for $\mathbf{u} \in \mathbf{H}(\operatorname{div}, K) \cap \mathbf{H}^r(K)^3$ with $r > 0$. Moreover, by virtue of (3.9) and (3.11), $\widehat{\mathbf{u}}_{\widehat{K}} = \widehat{\mathbf{u}_K}$,

in principle when $\mathbf{u} \in \mathbf{H}^r(K)^3$ with $r > \frac{1}{2}$, but, because of a density argument, the equality holds for $\mathbf{u} \in \mathbf{H}(\text{div}, K) \cap \mathbf{H}^r(K)^3$ with $r > 0$, too. Thus, (3.1) and (3.2) yield

$$\|\mathbf{u} - \mathbf{u}_K\|_{0,K}^2 \leq \int_{\hat{K}} \frac{|\widehat{D}\mathbf{F}_K|^2}{J_K} |\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_{\hat{K}}|^2 \leq \frac{\bar{\sigma}_K^2}{\sigma_K^3 h_K} \|\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_{\hat{K}}\|_{0,\hat{K}}^2.$$

The mapping $\widehat{\mathbf{u}} \mapsto (\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_{\hat{K}})$ defines a bounded linear operator from $\mathbf{H}(\widehat{\text{div}}, \hat{K}) \cap \mathbf{H}^r(\hat{K})^3$ into $\mathbf{L}^2(\hat{K})^3$ that vanishes on $\widehat{\mathcal{R}} \supset \mathcal{P}_0(\hat{K})^3$, where $\mathcal{P}_k$ denotes the space of polynomial functions of degree k , $k \in \mathbb{N}$. Hence, applying the Bramble-Hilbert lemma conveniently adapted to non-integer Sobolev spaces (see Dupont & Scott, 1980, for a proof of this lemma based on a suitable version of Deny-Lions Lemma in fractional spaces), we obtain

$$\|\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_{\hat{K}}\|_{0,\hat{K}} \leq C \left(|\widehat{\mathbf{u}}|_{r,\hat{K}} + \|\widehat{\text{div}} \widehat{\mathbf{u}}\|_{0,\hat{K}} \right);$$

(see Alonso & Valli, 1999, for a similar estimate for Nédélec edge elements). Therefore, the first estimate of the lemma follows from the last four inequalities.

To prove the second estimate, we use (3.10) to write

$$\|\text{div}(\mathbf{u} - \mathbf{u}_K)\|_{0,K} \leq \frac{1}{(\min_{\hat{K}} J_K)^{\frac{1}{2}}} \|\widehat{\text{div}}(\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_{\hat{K}})\|_{0,\hat{K}}.$$

Next, we proceed as in the proof of the first estimate and use (3.10) again to obtain

$$|\widehat{\text{div}} \widehat{\mathbf{u}}|_{q,\hat{K}} \leq \frac{\max_{\hat{K}} |\widehat{\nabla} J_K|}{(\min_{\hat{K}} J_K)^{\frac{1}{2}}} \|\text{div} \mathbf{u}\|_{0,K} + \frac{\max_{\hat{K}} J_K}{\min_{\hat{K}} J_K} \max_{\hat{K}} |\widehat{D}\mathbf{F}_K|^{\frac{3}{2}+q} |\text{div} \mathbf{u}|_{q,K}.$$

Hence, $\widehat{\text{div}} \widehat{\mathbf{u}} \in \mathbf{H}^q(\hat{K})$. On the other hand, $\widehat{\text{div}} \widehat{\mathbf{u}}_{\hat{K}} \in \mathcal{P}_0(\hat{K})$. Moreover, from the Gauss theorem and (3.11) (combined with a density argument if $r \leq \frac{1}{2}$), we have that $\widehat{\text{div}} \widehat{\mathbf{u}}_{\hat{K}}$ is the $\mathbf{L}^2$ projection of $\widehat{\text{div}} \widehat{\mathbf{u}}$ onto $\mathcal{P}_0(\hat{K})$. Consequently, using the Bramble-Hilbert lemma again, we obtain

$$\|\widehat{\text{div}}(\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_{\hat{K}})\|_{0,\hat{K}} \leq C |\widehat{\text{div}} \widehat{\mathbf{u}}|_{q,\hat{K}}.$$

Therefore, the second estimate of the lemma follows from the last three inequalities, (3.1), (3.2), and (3.7). Thus, we conclude the proof. $\square$

Remark 3.4 *If $\mathbf{u} \in \mathbf{H}^r(K)^3$ with $r > \frac{1}{2}$, then the first estimate of the lemma can be improved as follows:*

$$\|\mathbf{u} - \mathbf{u}_K\|_{0,K} \leq C \left(\frac{\varepsilon_K}{h_K} \|\mathbf{u}\|_{0,K} + h_K^r |\mathbf{u}|_{r,K} \right).$$

Indeed, for $r > \frac{1}{2}$, $\widehat{\mathbf{u}} \mapsto (\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_{\hat{K}})$ defines a bounded linear operator from $\mathbf{H}^r(\hat{K})^3$ into $\mathbf{L}^2(\hat{K})^3$ and Bramble-Hilbert lemma yields in this case $\|\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_{\hat{K}}\|_{0,\hat{K}} \leq C |\widehat{\mathbf{u}}|_{r,\hat{K}}$.

Remark 3.5 Girault & Raviart (1986, Theorem III.4.4) have extended some arguments from Thomas (1977, Theorem III.4.3) to prove the following improved estimate in 2D for $\mathbf{u} \in H^1(K)^3$:

$$\|\mathbf{u} - \mathbf{u}_K\|_{0,K} \leq Ch_K |\mathbf{u}|_{1,K}.$$

More recently, Arnold et al. (2002b, Theorem 7) obtained a similar result valid for 2D Raviart-Thomas elements of arbitrary order. However, the arguments of these references do not seem to hold in 3D. Indeed, both rely on the fact that, in 2D, the columns of the matrix $\text{Cof}(\widehat{D}\mathbf{F}_K)^t$ belong to $\mathcal{P}_{1,0} \times \mathcal{P}_{0,1}$, which, in this case, corresponds to $\widehat{\mathcal{R}}$ (see Arnold et al., 2002b, Lemma 9). Instead, in 3D, these columns belong to $\mathcal{P}_{2,1,1} \times \mathcal{P}_{1,2,1} \times \mathcal{P}_{1,1,2} \neq \widehat{\mathcal{R}}$.

The following lemma, which will be used in the sequel, shows that the Raviart-Thomas interpolant of a divergence-free vector field is also divergence-free:

Lemma 3.6 For all $\mathbf{u} \in H^r(K)^3$, $r > 0$, if $\text{div } \mathbf{u} = 0$ in K , then $\text{div } \mathbf{u}_K = 0$ in K , too.

Proof. Let $\mathbf{u} \in H^r(K)^3$, $r > 0$, such that $\text{div } \mathbf{u} = 0$ in K . According to (3.10) $\widehat{\text{div}} \widehat{\mathbf{u}} = 0$ in $\widehat{K}$ and

$$(\text{div } \mathbf{u}_K) \circ \mathbf{F}_K = \frac{1}{J_K} \widehat{\text{div}} \widehat{\mathbf{u}}_K = \frac{1}{J_K} \widehat{\text{div}} \widehat{\mathbf{u}}_{\widehat{K}},$$

the last equality because $\widehat{\mathbf{u}} \in H(\widehat{\text{div}}, \widehat{K}) \cap H^r(\widehat{K})^3$ and $\widehat{\mathbf{u}}_{\widehat{K}} = \widehat{\mathbf{u}}_K$, as shown in the proof of Lemma 3.3. Moreover, it was shown in the same proof that $\widehat{\text{div}} \widehat{\mathbf{u}}_{\widehat{K}}$ is the L^2 projection of $\widehat{\text{div}} \widehat{\mathbf{u}}$ onto $\mathcal{P}_0(\widehat{K})$. Then $\widehat{\text{div}} \widehat{\mathbf{u}}_{\widehat{K}} = 0$ in $\widehat{K}$ and, consequently, $\text{div } \mathbf{u}_K = 0$ in K , too. Thus, we conclude the proof. $\square$

3.3 Discrete kinematically admissible fluid-solid displacements

We consider two regular families of partitions in hexahedra of Ω_F and Ω_S , $\{\mathcal{T}_h^F\}$ and $\{\mathcal{T}_h^S\}$, respectively; $h := \max_{K \in \mathcal{T}_h^F \cup \mathcal{T}_h^S} h_K$ denotes the common mesh-size. We recall that a family of hexahedral meshes is said to be *regular*, if there exist two strictly positive constants σ_1 and σ_2 , the same for all meshes, such that for all elements K of each mesh, the shape-regularity constants $\bar{\sigma}_K$ and $\underline{\sigma}_K$ in (3.1) satisfy

$$\bar{\sigma}_K \leq \sigma_1 \quad \text{and} \quad \underline{\sigma}_K \geq \sigma_2. \quad (3.12)$$

We do not assume that $\{\mathcal{T}_h^F\}$ and $\{\mathcal{T}_h^S\}$ match on the fluid-solid interface. Therefore, the two meshes may induce different partitions on Γ_I . We denote by $\mathcal{T}_h^I$ the partition induced by $\mathcal{T}_h^F$:

$$\mathcal{T}_h^I := \{F \text{ is a face of a hexahedron } K \in \mathcal{T}_h^F : F \subset \Gamma_I\}.$$

However, the choice of meshes in the two domains is not completely free. In fact, we restrict our analysis to couples of meshes $\mathcal{T}_h^F$ and $\mathcal{T}_h^S$ such that each hexahedra in the fluid domain in contact with Γ_I is not much smaller than the neighboring hexahedra in the solid domain. More precisely, we assume that there exists a strictly positive constant σ_3 , the

same for all pairs of meshes, such that $\forall F \in \mathcal{T}_h^I$ and $\forall K \in \mathcal{T}_h^S$ such that $F \cap \partial K \neq \emptyset$, there holds

$$|F|^{\frac{1}{2}} \geq \sigma_3 h_K, \quad (3.13)$$

where $|F|$ denotes the 2D-measure of the face F .

We further assume that $\{\mathcal{T}_h^F\}$ is a family of *asymptotically parallelepiped* meshes, in the sense that for each element $K \in \mathcal{T}_h^F$ there exists a parallelepiped such that the distances of the vertices of K to those of the parallelepiped are $\mathcal{O}(h_K^2)$. More precisely, we assume that there exists a strictly positive constant σ_4 , the same for all meshes, such that constants ε_K in (3.6) satisfy

$$\varepsilon_K \leq \sigma_4 h_K^2 \quad \forall K \in \mathcal{T}_h^F. \quad (3.14)$$

Notice that for any family of asymptotically parallelepiped meshes, as a consequence of Lemma 3.1, the estimates for the derivatives of J_K and the entries of $\text{Cof}(\widehat{D}\mathbf{F}_K)$ are one order higher than the corresponding ones for J_K and $\text{Cof}(\widehat{D}\mathbf{F}_K)$ given by (3.2) and (3.3), respectively.

We use lowest-order Raviart-Thomas hexahedral finite elements to discretize the fluid displacements:

$$\mathcal{R}_h := \{\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega_F) : \mathbf{u}_h|_K \in \mathcal{R}_K \quad \forall K \in \mathcal{T}_h^F\}.$$

Each component of the solid displacement field is discretized by standard Lagrangian trilinear isoparametric hexahedral elements:

$$\mathcal{L}_h := \left\{ \mathbf{w}_h \in \mathbf{H}_{\Gamma_D}^1(\Omega_S)^3 : \mathbf{w}_h \circ \mathbf{F}_K \in \mathcal{Q}_1(\widehat{K})^3 \quad \forall K \in \mathcal{T}_h^S \right\},$$

where $\mathcal{Q}_k(\widehat{K}) := \mathcal{P}_{k,k,k}(\widehat{K})$, $k \in \mathbb{N}$.

Finally, the kinematic constraint (2.5) must be imposed somehow to the discrete fluid-solid displacements. Since doing it strongly (i.e., imposing $\mathbf{u}_h \cdot \mathbf{n} = \mathbf{w}_h \cdot \mathbf{n}$ on Γ_1) would be too stringent (see Bermúdez & Rodríguez, 1994) we impose it weakly as follows:

$$\int_F \mathbf{u}_h \cdot \mathbf{n} = \int_F \mathbf{w}_h \cdot \mathbf{n} \quad \forall F \in \mathcal{T}_h^I. \quad (3.15)$$

This constraint allows us to eliminate by static condensation the degrees of freedom of the fluid displacement corresponding to these faces.

Thus, the space of kinematically admissible discrete displacements is the following one:

$$\mathcal{V}_h := \{(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{R}_h \times \mathcal{L}_h : (3.15) \text{ holds true}\}.$$

3.4 Approximation error estimates

Next, we present a result for the approximation of the continuous space $\mathcal{V}$ from the finite element space $\mathcal{V}_h$.

Raviart-Thomas elements are $\mathbf{H}(\text{div})$ -conforming in the following sense: Given $\mathbf{u} \in \mathbf{H}(\text{div}, \Omega_F) \cap \mathbf{H}^r(\Omega_F)^3$ with $r > 0$ (or $\mathbf{u} \in \mathbf{H}^r(\Omega_F)^3$ with $r > \frac{1}{2}$) if we define

$$\mathbf{u}^R \in \mathbf{L}^2(\Omega_F)^3 : \quad \mathbf{u}^R|_K := \mathbf{u}_K \quad \forall K \in \mathcal{T}_h^F,$$

then $\mathbf{u}^R \in H(\operatorname{div}, \Omega_F)$ and, consequently, $\mathbf{u}^R \in \mathcal{R}_h$. This is the so-called Raviart-Thomas interpolation operator:

$$\begin{aligned} H(\operatorname{div}, \Omega_F) \cap H^r(\Omega_F)^3 &\longrightarrow \mathcal{R}_h, \\ \mathbf{u} &\longmapsto \mathbf{u}^R. \end{aligned}$$

Let us recall that, because of (3.11), if $r > \frac{1}{2}$, then $\mathbf{u}^R$ is the unique vector field in $\mathcal{R}_h$ satisfying

$$\int_F \mathbf{u}^R \cdot \mathbf{n}_F = \int_F \mathbf{u} \cdot \mathbf{n}_F \quad (3.16)$$

for all faces F of all hexahedra $K \in \mathcal{T}_h^F$.

The following result is an immediate consequence of Lemma 3.3 and the assumption that $\{\mathcal{T}_h^F\}$ is a regular family of asymptotically parallelepiped meshes (c.f. (3.14)):

Corollary 3.7 *Let $\mathbf{u} \in H^r(\Omega_F)^3$ be such that $\operatorname{div} \mathbf{u} \in H^q(\Omega_F)$, with $0 < q, r \leq 1$. Let $\mathbf{u}^R \in \mathcal{R}_h$ be the Raviart-Thomas interpolant of $\mathbf{u}$. Then the following estimates hold true:*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}^R\|_{0, \Omega_F} &\leq Ch^r \left(\|\mathbf{u}\|_{r, \Omega_F} + \|\operatorname{div} \mathbf{u}\|_{0, \Omega_F} \right), \\ \|\operatorname{div} (\mathbf{u} - \mathbf{u}^R)\|_{0, \Omega_F} &\leq Ch^q \|\operatorname{div} \mathbf{u}\|_{q, \Omega_F}. \end{aligned}$$

Remark 3.8 *If $\mathbf{u} \in H^r(K)^3$ with $r > \frac{1}{2}$, then the first estimate above can be improved according to Remark 3.4 as follows:*

$$\|\mathbf{u} - \mathbf{u}^R\|_{0, \Omega_F} \leq Ch^r \|\mathbf{u}\|_{r, \Omega_F}.$$

Remark 3.9 *The power 2 of h_K in the definition (3.14) of asymptotically parallelepiped meshes is somewhat arbitrary. Indeed, the corollary and remark above hold true as long as $\varepsilon_K \leq Ch_K^{1+\max\{r, q\}}$.*

On the solid domain, we consider a Clment's type interpolation, which preserves vanishing traces on Γ_D (see Clment, 1975):

$$\begin{aligned} H_{\Gamma_D}^1(\Omega_S)^3 &\longrightarrow \mathcal{L}_h, \\ \mathbf{w} &\longmapsto \mathbf{w}^I. \end{aligned}$$

This interpolant, which is based on local averages, satisfies

$$\|\mathbf{w} - \mathbf{w}^I\|_{m, K} \leq Ch_K^{l-m} \|\mathbf{w}\|_{l, \tilde{K}} \quad \forall \mathbf{w} \in H^l(\tilde{K}), \quad 0 \leq m \leq l \leq 2, \quad (3.17)$$

where $\tilde{K} := \bigcup \{K' \in \mathcal{T}_h^S : K' \text{ and } K \text{ share a vertex}\}$ and C only depends on the shape-regularity constants σ_1 and σ_2 . This estimate and standard interpolation results on Sobolev spaces yield the following approximation result:

Lemma 3.10 *Let $\mathbf{w} \in H_{\Gamma_D}^1(\Omega_S)^3 \cap H^{1+s}(\Omega_S)^3$, with $0 \leq s \leq 1$. Then the Clment's interpolant $\mathbf{w}^I$ satisfies*

$$\|\mathbf{w} - \mathbf{w}^I\|_{1, \Omega_S} \leq Ch^s \|\mathbf{w}\|_{1+s, \Omega_S}.$$

In general, $(\mathbf{u}^R, \mathbf{w}^I) \notin \mathcal{V}_h$. Then, to obtain a $\mathcal{V}_h$ -interpolant of $(\mathbf{u}, \mathbf{w})$, we modify the degrees of freedom of $\mathbf{u}^R$ on the faces lying on Γ_1 . Thus, we define $\mathbf{u}^I$ as the unique vector field in $\mathcal{R}_h$ satisfying

$$\int_F \mathbf{u}^I \cdot \mathbf{n}_F = \begin{cases} \int_F \mathbf{u}^R \cdot \mathbf{n}_F, & \text{if } F \not\subset \Gamma_1, \\ \int_F \mathbf{w}^I \cdot \mathbf{n}, & \text{if } F \subset \Gamma_1. \end{cases}$$

The following approximation result holds for this interpolation:

Lemma 3.11 *Let $(\mathbf{u}, \mathbf{w}) \in \mathcal{V}$ be such that $\mathbf{u} \in H^r(\Omega_F)^3$, $\operatorname{div} \mathbf{u} \in H^q(\Omega_F)$, and $\mathbf{w} \in H^{1+s}(\Omega_S)^3$, with $0 < q, r, s \leq 1$. Let $(\mathbf{u}^I, \mathbf{w}^I) \in \mathcal{V}_h$ be as defined above. Then the following estimates hold true:*

$$\begin{aligned} \|\mathbf{u}^I - \mathbf{u}\|_{\operatorname{div}, \Omega_F} &\leq Ch^{\min\{q, r, s\}} \left(\|\mathbf{u}\|_{r, \Omega_F} + \|\operatorname{div} \mathbf{u}\|_{q, \Omega_F} + \|\mathbf{w}\|_{1+s, \Omega_S} \right), \\ \|\mathbf{w}^I - \mathbf{w}\|_{1, \Omega_S} &\leq Ch^s \|\mathbf{w}\|_{1+s, \Omega_S}. \end{aligned}$$

Proof. By virtue of Lemma 3.10 we only have to prove the first estimate. Let us write

$$\|\mathbf{u} - \mathbf{u}^I\|_{\operatorname{div}, \Omega_F} \leq \|\mathbf{u} - \mathbf{u}^R\|_{\operatorname{div}, \Omega_F} + \|\mathbf{u}^R - \mathbf{u}^I\|_{\operatorname{div}, \Omega_F}.$$

A bound for the first term on the right-hand side above is given in Corollary 3.7. To estimate the second one, from the definitions of $\mathbf{u}^I$ and $\mathbf{u}^R$ and since $\mathbf{u} \cdot \mathbf{n} = \mathbf{w} \cdot \mathbf{n}$ on Γ_1 , we have

$$\mathbf{u}^I - \mathbf{u}^R = \sum_{F \in \mathcal{T}_h^I} \left[\frac{1}{|F|} \int_F (\mathbf{w}^I - \mathbf{u}^R) \cdot \mathbf{n} \right] \phi_F = \sum_{F \in \mathcal{T}_h^I} \left[\frac{1}{|F|} \int_F (\mathbf{w}^I - \mathbf{w}) \cdot \mathbf{n} \right] \phi_F,$$

where ϕ_F is the Raviart-Thomas basis function associated with F ; namely, $\phi_F \in \mathcal{R}_h$: $\frac{1}{|F|} \int_F \phi_F \cdot \mathbf{n}_F = 1$ and $\frac{1}{|F'|} \int_{F'} \phi_F \cdot \mathbf{n}_{F'} = 0$ for all other faces $F' \neq F$. Consequently,

$$\|\mathbf{u}^I - \mathbf{u}^R\|_{\operatorname{div}, \Omega_F}^2 = \sum_{F \in \mathcal{T}_h^I} \left| \frac{1}{|F|} \int_F (\mathbf{w}^I - \mathbf{w}) \cdot \mathbf{n} \right|^2 \|\phi_F\|_{\operatorname{div}, \Omega_F}^2. \quad (3.18)$$

Given a face $F \in \mathcal{T}_h^I$, let $K \in \mathcal{T}_h^F$ be the hexahedron such that F is a face of K . Let $\hat{\phi}_F$ be the Piola transform of ϕ_F ; namely, $\hat{\phi}_F = J_K (\hat{D} \mathbf{F}_K)^{-1} (\phi_F \circ \mathbf{F}_K)$. Then

$$\|\phi_F\|_{0, K}^2 = \int_{\hat{K}} \frac{1}{J_K} |\hat{D} \mathbf{F}_K \hat{\phi}_F|^2 \leq \frac{\max_{\hat{K}} |\hat{D} \mathbf{F}_K|^2}{\min_{\hat{K}} J_K} \int_{\hat{K}} |\hat{\phi}_F|^2 \leq \frac{\sigma_1^2}{\sigma_2^3 h_K} \int_{\hat{K}} |\hat{\phi}_F|^2,$$

the latter because of (3.1), (3.2), and (3.12). Now, from (3.9), we have $\int_{\hat{F}'} \hat{\phi}_F \cdot \mathbf{n}_{\hat{F}'} = \int_{F'} \phi_F \cdot \mathbf{n}_{F'}$ for all faces $\hat{F}'$ of $\hat{K}$. Hence, $\hat{\phi}_F = |F| \phi_{\hat{F}}$, with $\phi_{\hat{F}}$ being the basis function of $\hat{\mathcal{R}}$ associated with $\hat{F}$. Consequently, straightforward computations lead to $\int_{\hat{K}} |\hat{\phi}_F|^2 = \frac{1}{3} |F|^2$ and thus

$$\|\phi_F\|_{0, K}^2 \leq \frac{\sigma_1^2 |F|^2}{3\sigma_2^3 h_K}.$$

Furthermore, from (3.10) we have

$$\|\operatorname{div} \phi_F\|_{0,K}^2 = \int_{\widehat{K}} \frac{1}{J_K} |\widehat{\operatorname{div}} \widehat{\phi}_F|^2.$$

Since $\widehat{\operatorname{div}} \widehat{\phi}_F$ is constant, by applying the Gauss theorem and taking into account (3.9) we obtain

$$\widehat{\operatorname{div}} \widehat{\phi}_F = \int_{\widehat{K}} \widehat{\operatorname{div}} \widehat{\phi}_F = \int_{\partial \widehat{K}} \widehat{\phi}_F \cdot \mathbf{n}_{\partial \widehat{K}} = \int_F \phi_F \cdot \mathbf{n}_F = |F|.$$

Thus, (3.2) yields

$$\|\operatorname{div} \phi_F\|_{0,K}^2 = |F|^2 \int_{\widehat{K}} \frac{1}{J_K} \leq \frac{|F|^2}{\sigma_2^3 h_K^3}.$$

As a consequence of the shape-regularity property (3.12), it is well known that

$$C_1 h_K^2 \leq |F| \leq C_2 h_K^2$$

with C_1 and C_2 strictly positive constants depending only on σ_1 and σ_2 (see also the proof of Lemma 4.9 below). Therefore, the last three inequalities yield

$$\|\phi_F\|_{\operatorname{div}, \Omega_F}^2 = \|\phi_F\|_{\operatorname{div}, K}^2 \leq C |F|^{\frac{1}{2}}.$$

By using this and the Cauchy-Schwarz inequality to bound (3.18), we obtain

$$\|\mathbf{u}^I - \mathbf{u}^R\|_{\operatorname{div}, \Omega_F}^2 \leq C \sum_{F \in \mathcal{T}_h^I} \frac{1}{|F|^{\frac{1}{2}}} \int_F |\mathbf{w}^I - \mathbf{w}|^2.$$

Because of the assumption (3.13), the sum above can be bounded as follows:

$$\sum_{F \in \mathcal{T}_h^I} \frac{1}{|F|^{\frac{1}{2}}} \int_F |\mathbf{w}^I - \mathbf{w}|^2 \leq \sum_{K' \in \mathcal{T}_h^S: \partial K' \cap \Gamma_1 \neq \emptyset} \frac{1}{\sigma_3 h_{K'}} \int_{\partial K' \cap \Gamma_1} |\mathbf{w}^I - \mathbf{w}|^2,$$

while for each $K' \in \mathcal{T}_h^S$ such that $\partial K' \cap \Gamma_1 \neq \emptyset$, the trace theorem and a scaling argument yield

$$\int_{\partial K' \cap \Gamma_1} |\mathbf{w}^I - \mathbf{w}|^2 \leq C \left(\frac{1}{h_{K'}} \|\mathbf{w}^I - \mathbf{w}\|_{0,K'}^2 + h_{K'} \|\mathbf{w}^I - \mathbf{w}\|_{1,K'}^2 \right).$$

Therefore, if $\mathbf{w} \in H^l(\Omega_S)$, $l = 1, 2$, we use the last three inequalities and (3.17) to obtain

$$\|\mathbf{u}^I - \mathbf{u}^R\|_{\operatorname{div}, \Omega_F}^2 \leq C \sum_{K' \in \mathcal{T}_h^S: \partial K' \cap \Gamma_1 \neq \emptyset} h_{K'}^{2l-2} \|\mathbf{w}\|_{l, \tilde{K}'}^2 \leq C h^{2l-2} \|\mathbf{w}\|_{l, \Omega_S}^2, \quad l = 1, 2.$$

Hence, as a consequence of standard results on interpolation between Sobolev spaces, we finally have

$$\|\mathbf{u}^I - \mathbf{u}^R\|_{\operatorname{div}, \Omega_F} \leq C h^s \|\mathbf{w}\|_{1+s, \Omega_S}$$

and we conclude the proof. $\square$

Remark 3.12 Note that the above lemma is valid even though the meshes on the fluid and solid domains do not match on the fluid-solid interface Γ_1 .

4 Finite Element Approximation

Our next step is to define the discrete spectral problem:

Problem 4.1 Find $\lambda_h \in \mathbb{R}$ and $(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{V}_h$, $(\mathbf{u}_h, \mathbf{w}_h) \neq (\mathbf{0}, \mathbf{0})$, such that

$$a((\mathbf{u}_h, \mathbf{w}_h), (\mathbf{v}_h, \mathbf{z}_h)) = \lambda_h b((\mathbf{u}_h, \mathbf{w}_h), (\mathbf{v}_h, \mathbf{z}_h)) \quad \forall (\mathbf{v}_h, \mathbf{z}_h) \in \mathcal{V}_h.$$

This is a non-conforming approximation of Problem 2.2 because, for $(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{V}_h$, in general $\mathbf{u}_h \cdot \mathbf{n} \neq \mathbf{w}_h \cdot \mathbf{n}$ on Γ_i and, therefore, $\mathcal{V}_h \not\subset \mathcal{V}$.

4.1 Spectral characterization of the discrete problem

Clearly $\lambda_h = 0$ is an eigenvalue of Problem 4.1 with corresponding eigenspace

$$\mathcal{K}_h := \{(\mathbf{u}_h, \mathbf{0}) \in \mathcal{V}_h : \operatorname{div} \mathbf{u}_h = 0 \text{ in } \Omega_F\}.$$

Although the approximation is non-conforming we have the following lemma:

Lemma 4.2 The following inclusion holds true:

$$\mathcal{K}_h \subset \mathcal{K}.$$

Proof. Let $(\mathbf{u}_h, \mathbf{0}) \in \mathcal{K}_h$. Since $\operatorname{div} \mathbf{u}_h = 0$ in Ω_F , we only have to prove that $\mathbf{u}_h \cdot \mathbf{n} = 0$ on Γ_i . Now, because of (3.9) and (3.15), $\int_{\widehat{F}} \widehat{\mathbf{u}}_h \cdot \mathbf{n}_{\widehat{F}} = \int_F \mathbf{u}_h \cdot \mathbf{n}_F = 0 \quad \forall F \in \mathcal{T}_h^1$. Hence, since $\widehat{\mathbf{u}}_h \cdot \mathbf{n}_{\widehat{F}}$ is constant on $\widehat{F}$, $\widehat{\mathbf{u}}_h \cdot \mathbf{n}_{\widehat{F}}|_{\widehat{F}} = 0$. Therefore, from (3.9) again, $\int_S \mathbf{u}_h \cdot \mathbf{n}_S = \int_{\widehat{S}} \widehat{\mathbf{u}}_h \cdot \mathbf{n}_{\widehat{S}} = 0 \quad \forall S \subset F$ and, consequently, $\mathbf{u}_h \cdot \mathbf{n}_F|_F = 0$, too. As this holds $\forall F \in \mathcal{T}_h^1$, we have proved that $(\mathbf{u}_h, \mathbf{0}) \in \mathcal{K}$. $\square$

Let $\mathcal{G}_h$ be the orthogonal complement of $\mathcal{K}_h$ in $\mathcal{V}_h$ with respect to the inner product b . Clearly, $\mathcal{K}_h$ and $\mathcal{G}_h$ are orthogonal in the Hilbert products of $\mathcal{H}$ and $\mathcal{X}$, too. Notice that, in general, $\mathcal{G}_h \not\subset \mathcal{G}$. The following lemma establishes a sort of Helmholtz decomposition of $\mathcal{G}_h$:

Lemma 4.3 For all $(\mathbf{v}_h, \mathbf{z}_h) \in \mathcal{G}_h$ there exists $\xi \in H^1(\Omega_F)$ and $\chi \in H(\operatorname{div}, \Omega_F)$ such that $(\nabla \xi, \mathbf{z}_h) \in \mathcal{G}$, $\operatorname{div} \chi = 0$ in Ω_F , and

$$\mathbf{v}_h = \nabla \xi + \chi.$$

Moreover, there exist constants $r \in (\frac{1}{2}, 1]$ and $C > 0$ such that $\nabla \xi \in H^r(\Omega_F)^3$ and

$$\|\nabla \xi\|_{r, \Omega_F} \leq C \left(\|\operatorname{div} \mathbf{v}_h\|_{0, \Omega_F} + \|\mathbf{z}_h\|_{1, \Omega_S} \right), \quad (4.1)$$

$$\|\chi\|_{0, \Omega_F} \leq Ch^r \left(\|\operatorname{div} \mathbf{v}_h\|_{0, \Omega_F} + \|\mathbf{z}_h\|_{1, \Omega_S} \right). \quad (4.2)$$

Proof. Given $(\mathbf{v}_h, \mathbf{z}_h) \in \mathcal{G}_h$, let $\xi \in H^1(\Omega_F)$ be a solution of the Neumann problem

$$\begin{cases} \Delta \xi = \operatorname{div} \mathbf{v}_h & \text{in } \Omega_F, \\ \frac{\partial \xi}{\partial \mathbf{n}} = \mathbf{z}_h \cdot \mathbf{n} & \text{on } \Gamma_i. \end{cases}$$

Notice that this problem is compatible. Indeed, from the Gauss theorem and (3.15), $\int_{\Omega_F} \operatorname{div} \mathbf{v}_h = \int_{\Gamma_1} \mathbf{v}_h \cdot \mathbf{n} = \int_{\Gamma_1} \mathbf{z}_h \cdot \mathbf{n}$. Furthermore, from the standard *a priori* estimate (see Dauge, 1988) $\exists r \in (\frac{1}{2}, 1]$ such that $\nabla \xi \in H^r(\Omega_F)$ and (4.1) holds true.

Moreover, clearly $(\nabla \xi, \mathbf{z}_h) \in \mathcal{G}$ and, if we define $\chi := \mathbf{v}_h - \nabla \xi$, then $\operatorname{div} \chi = 0$ in Ω_F . Thus, there only remains to prove (4.2).

For this purpose, we write

$$\|\chi\|_{0,\Omega_F}^2 = \int_{\Omega_F} \chi \cdot (\mathbf{v}_h - \nabla \xi) = \int_{\Omega_F} \chi \cdot [\mathbf{v}_h - (\nabla \xi)^R] + \int_{\Omega_F} \chi \cdot [(\nabla \xi)^R - \nabla \xi]. \quad (4.3)$$

Now, $\mathbf{v}_h - (\nabla \xi)^R = (\mathbf{v}_h - \nabla \xi)^R$ and $\operatorname{div}(\mathbf{v}_h - \nabla \xi) = 0$ in Ω_F . Therefore, because of Lemma 3.6, $\operatorname{div}[\mathbf{v}_h - (\nabla \xi)^R] = 0$ in Ω_F , too. On the other hand, from (3.16), $\forall F \in \mathcal{T}_h^1$ we have

$$\int_F [\mathbf{v}_h - (\nabla \xi)^R] \cdot \mathbf{n} = \int_F (\mathbf{v}_h - \nabla \xi) \cdot \mathbf{n} = \int_F (\mathbf{v}_h - \mathbf{z}_h) \cdot \mathbf{n} = 0.$$

Consequently, $(\mathbf{v}_h - (\nabla \xi)^R, \mathbf{0})$ satisfies (3.15) and, hence, it belongs to $\mathcal{K}_h$. Thus, from Lemma 4.2, $(\mathbf{v}_h - (\nabla \xi)^R, \mathbf{0}) \in \mathcal{K}$, too. Therefore,

$$\int_{\Omega_F} \chi \cdot [\mathbf{v}_h - (\nabla \xi)^R] = \int_{\Omega_F} \mathbf{v}_h \cdot [\mathbf{v}_h - (\nabla \xi)^R] - \int_{\Omega_F} \nabla \xi \cdot [\mathbf{v}_h - (\nabla \xi)^R] = 0, \quad (4.4)$$

because $(\mathbf{v}_h, \mathbf{z}_h) \in \mathcal{G}_h \perp \mathcal{K}_h$ and $(\nabla \xi, \mathbf{z}_h) \in \mathcal{G} \perp \mathcal{K}$.

To estimate the other term in the right-hand side of (4.3), we use that $\nabla \xi \in H^r(\Omega_F)$ with $r > 1/2$. Hence, Remark 3.8 and the Cauchy-Schwarz inequality yield

$$\int_{\Omega_F} \chi \cdot [(\nabla \xi)^R - \nabla \xi] \leq Ch^r \|\chi\|_{0,\Omega_F} \|\nabla \xi\|_{r,\Omega_F}. \quad (4.5)$$

Finally, (4.2) follows from (4.3), (4.4), (4.5), and (4.1). Thus we conclude the proof. $\square$

The previous lemma implies that the bilinear form a is $\mathcal{K}$ -elliptic on $\mathcal{G}_h$. Thus, we are able to define an operator $T_h : \mathcal{H} \rightarrow \mathcal{G}_h$, which plays in the discrete problem the same role as T in the continuous one: Given $(\mathbf{f}, \mathbf{g}) \in \mathcal{H}$, let $T_h(\mathbf{f}, \mathbf{g}) := (\mathbf{u}_h, \mathbf{w}_h)$, with $(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{G}_h$ being the solution of

$$a((\mathbf{u}_h, \mathbf{w}_h), (\mathbf{v}_h, \mathbf{z}_h)) = b((\mathbf{f}, \mathbf{g}), (\mathbf{v}_h, \mathbf{z}_h)) \quad \forall (\mathbf{v}_h, \mathbf{z}_h) \in \mathcal{G}_h.$$

The linear operator T_h is self-adjoint with respect to a and b . Thus, its eigenvalues μ_h are real and positive. Moreover, as in the continuous case, λ_h is a positive eigenvalue of Problem 4.1 if and only if $\mu_h = 1/\lambda_h$ is a positive eigenvalue of the operator T_h , and the corresponding associated eigenfunctions coincide.

4.2 Spectral approximation

Our next goal is to show that the eigenvalues and eigenfunctions of Problem 4.1 converge to those of Problem 2.2 without spectral pollution. We will do this by using the standard

spectral approximation theory (see Babuška & Osborn, 1991). For this purpose, we show first that the operators T_h converge to T in norm as h goes to zero, by adapting to our case some results from Durán *et al.* (2000).

From now on, let r and s denote the constants appearing in Theorem 2.4 and let $t := \min\{r, s\}$. Let us remark that constant r in Lemma 4.3 coincides with that of Theorem 2.4. Indeed, in the two cases, r is the regularity constant of the Neumann problem for the Laplace operator in Ω_F (see Remark 2.3 and the proof of Lemma 4.3).

The following two lemmas give estimates of the consistency and approximation errors for our non-conforming method.

Lemma 4.4 *Given $(\mathbf{f}, \mathbf{g}) \in \mathcal{H}$, let $(\mathbf{u}, \mathbf{w}) = T(\mathbf{f}, \mathbf{g})$ and $(\mathbf{u}_h, \mathbf{w}_h) = T_h(\mathbf{f}, \mathbf{g})$. Then, $\forall (\mathbf{v}_h, \mathbf{z}_h) \in \mathcal{G}_h$,*

$$|a((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), (\mathbf{v}_h, \mathbf{z}_h))| \leq Ch^r \|\mathbf{f}\|_{0, \Omega_F} \left(\|\operatorname{div} \mathbf{v}_h\|_{0, \Omega_F} + \|\mathbf{z}_h\|_{1, \Omega_S} \right).$$

Proof. Let $(\mathbf{v}_h, \mathbf{z}_h) \in \mathcal{G}_h$. We write $\mathbf{v}_h = \nabla \xi + \chi$, with $(\nabla \xi, \mathbf{z}_h) \in \mathcal{G}$ and $\operatorname{div} \chi = 0$ as in Lemma 4.3. Then

$$\begin{aligned} a((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), (\mathbf{v}_h, \mathbf{z}_h)) &= a((\mathbf{u}, \mathbf{w}), (\nabla \xi, \mathbf{z}_h)) - b((\mathbf{f}, \mathbf{g}), (\mathbf{v}_h, \mathbf{z}_h)) \\ &= -b((\mathbf{f}, \mathbf{g}), (\chi, \mathbf{0})) = - \int_{\Omega_F} \rho_F \mathbf{f} \cdot \chi, \end{aligned}$$

and the lemma follows from (4.2). $\square$

Lemma 4.5 *Let $(\mathbf{f}, \mathbf{g}) \in \mathcal{H}$ and $(\mathbf{u}, \mathbf{w}) = T(\mathbf{f}, \mathbf{g})$. There exists $(\tilde{\mathbf{u}}_h, \tilde{\mathbf{w}}_h) \in \mathcal{G}_h$ such that*

$$\|(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{w} - \tilde{\mathbf{w}}_h)\|_{\mathcal{X}} \leq Ch^t \|(\mathbf{f}, \mathbf{g})\|_{\mathcal{H}}.$$

Proof. For $(\mathbf{u}, \mathbf{w}) = T(\mathbf{f}, \mathbf{g})$, because of Theorem 2.4, $\mathbf{u} \in H^r(\Omega_F)^3$, $\operatorname{div} \mathbf{u} \in H^1(\Omega_F)$, and $\mathbf{w} \in H^{1+s}(\Omega_S)^3$, with $r > \frac{1}{2}$ and $s > 0$. Therefore, let $(\mathbf{u}^I, \mathbf{w}^I) \in \mathcal{V}_h$ be as in Lemma 3.11. Let $P : \mathcal{V}_h \rightarrow \mathcal{K}_h$ be the orthogonal projection onto $\mathcal{K}_h$ in the inner product b . Then $(\tilde{\mathbf{u}}_h, \tilde{\mathbf{w}}_h) := (\mathbf{u}^I, \mathbf{w}^I) - P(\mathbf{u}^I, \mathbf{w}^I) \in \mathcal{G}_h$. Moreover, since $P(\mathbf{u}^I, \mathbf{w}^I) \in \mathcal{K}_h \subset \mathcal{K}$, $P(\mathbf{u}^I, \mathbf{w}^I)$ is orthogonal in the Hilbert product of $\mathcal{X}$ to $(\tilde{\mathbf{u}}_h, \tilde{\mathbf{w}}_h) \in \mathcal{G}_h$ and to $(\mathbf{u}, \mathbf{w}) \in \mathcal{G}$. Hence,

$$\begin{aligned} \|(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{w} - \tilde{\mathbf{w}}_h)\|_{\mathcal{X}} &\leq \|(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{w} - \tilde{\mathbf{w}}_h) - P(\mathbf{u}^I, \mathbf{w}^I)\|_{\mathcal{X}} \\ &= \|(\mathbf{u} - \mathbf{u}^I, \mathbf{w} - \mathbf{w}^I)\|_{\mathcal{X}} \leq Ch^t \|(\mathbf{f}, \mathbf{g})\|_{\mathcal{H}}, \end{aligned}$$

the latter because of the estimates in Lemma 3.11 and Theorem 2.4. $\square$

Now we are able to prove the convergence in norm of T_h to T .

Lemma 4.6 *There exists a positive constant C such that, $\forall (\mathbf{f}, \mathbf{g}) \in \mathcal{H}$,*

$$\|(T - T_h)(\mathbf{f}, \mathbf{g})\|_{\mathcal{X}} \leq Ch^t \|(\mathbf{f}, \mathbf{g})\|_{\mathcal{H}}.$$

Proof. Let $(\mathbf{u}, \mathbf{w}) := T(\mathbf{f}, \mathbf{g})$ and $(\mathbf{u}_h, \mathbf{w}_h) := T_h(\mathbf{f}, \mathbf{g})$. Since a is $\mathcal{X}$ -elliptic on $\mathcal{G}_h$, according to Strang's lemma (see for instance Ciarlet, 1991)

$$\begin{aligned} \|(\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h)\|_{\mathcal{X}} &\leq \inf_{(\tilde{\mathbf{u}}_h, \tilde{\mathbf{w}}_h) \in \mathcal{G}_h} \|(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{w} - \tilde{\mathbf{w}}_h)\|_{\mathcal{X}} \\ &\quad + C \sup_{(\mathbf{0}, \mathbf{0}) \neq (\mathbf{v}_h, \mathbf{z}_h) \in \mathcal{G}_h} \frac{a((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), (\mathbf{v}_h, \mathbf{z}_h))}{\|(\mathbf{v}_h, \mathbf{z}_h)\|_{\mathcal{X}}} \\ &\leq Ch^t \|(\mathbf{f}, \mathbf{g})\|_{\mathcal{H}}, \end{aligned}$$

the latter because of Lemmas 4.4 and 4.5. Thus we conclude the proof. $\square$

As a consequence of the above lemma and the continuity of the inclusion $\mathcal{X} \hookrightarrow \mathcal{H}$, $\|T - T_h\|_{\mathcal{X}}$ and $\|T - T_h\|_{\mathcal{H}}$, both converge to zero when h goes to zero.

Let $\mu > 0$ be an eigenvalue of T with multiplicity m and let $\mathcal{E}$ be its associated eigenspace. Since $\|T - T_h\|_{\mathcal{X}}$ converges to zero when h tends to zero, there exist m eigenvalues of T_h (repeated according to their respective multiplicities) converging to μ (see Kato, 1976). Let $\mathcal{E}_h$ be the direct sum of their corresponding associated eigenspaces.

We remind the definition of the *gap* $\hat{\delta}$ between two closed subspaces, $\mathcal{Y}$ and $\mathcal{Z}$, of $\mathcal{X}$:

$$\hat{\delta}(\mathcal{Y}, \mathcal{Z}) := \max \{ \delta(\mathcal{Y}, \mathcal{Z}), \delta(\mathcal{Z}, \mathcal{Y}) \},$$

with

$$\delta(\mathcal{Y}, \mathcal{Z}) := \sup_{\substack{(\mathbf{u}, \mathbf{w}) \in \mathcal{Y} \\ \|(\mathbf{u}, \mathbf{w})\|_{\mathcal{X}} = 1}} \left[\inf_{(\hat{\mathbf{u}}, \hat{\mathbf{w}}) \in \mathcal{Z}} \|(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{w} - \hat{\mathbf{w}})\|_{\mathcal{X}} \right].$$

The following theorem implies an optimal order of convergence for the approximation of the eigenfunctions.

Theorem 4.7 *There exists a strictly positive constant C such that*

$$\hat{\delta}(\mathcal{E}_h, \mathcal{E}) \leq Ch^t.$$

Proof. According to Theorem 7.1 from Babuška & Osborn (1991),

$$\hat{\delta}(\mathcal{E}_h, \mathcal{E}) \leq \|(T - T_h)|_{\mathcal{E}}\|_{\mathcal{X}},$$

whereas, because of Lemma 4.6 and the continuity of the inclusion $\mathcal{X} \hookrightarrow \mathcal{H}$,

$$\|(T - T_h)|_{\mathcal{E}}\|_{\mathcal{X}} := \sup_{(\mathbf{0}, \mathbf{0}) \neq (\mathbf{f}, \mathbf{g}) \in \mathcal{E}} \frac{\|(T - T_h)(\mathbf{f}, \mathbf{g})\|_{\mathcal{X}}}{\|(\mathbf{f}, \mathbf{g})\|_{\mathcal{X}}} \leq Ch^t.$$

Thus, we conclude the proof. $\square$

Remark 4.8 *Another consequence of the convergence in norm of T_h to T is that the numerical method does not introduce spurious modes. Indeed, given any compact set $K \subset \mathbb{C}$ not intersecting the spectrum of T , $\exists h_K > 0$ such that, $\forall h < h_K$, K does not intersect the spectrum of T_h , either (see Kato, 1976).*

4.3 Double-order approximation of the eigenvalues

The results from Rodríguez & Solomin (1996) can be adapted to our problem to prove an optimal order of convergence for the approximate eigenvalues in the 2D case. However, an alternative proof is necessary in 3D, because $\mathbf{u}_h \cdot \mathbf{n}|_{\Gamma_1}$ is not piecewise constant in general. To overcome this, we have the following result valid for asymptotically parallelepiped meshes. First, we introduce the subspace of piecewise constant functions on Γ_1 on the partition $\mathcal{T}_h^1$:

$$\mathcal{C}_h := \{ \psi_h \in L^2(\Gamma_1) : \psi_h|_F \in \mathcal{P}_0(F) \ \forall F \in \mathcal{T}_h^1 \},$$

and the L^2 orthogonal projection onto this space: $\Pi : L^2(\Gamma_1) \rightarrow \mathcal{C}_h$.

Lemma 4.9 *For all $(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{G}_h$,*

$$\|\Pi(\mathbf{u}_h \cdot \mathbf{n}) - \mathbf{u}_h \cdot \mathbf{n}\|_{0,\Gamma_1} \leq Ch \|\mathbf{w}_h \cdot \mathbf{n}\|_{0,\Gamma_1}.$$

Proof. Let $(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{G}_h$. Let $F \in \mathcal{T}_h^1$. Explicit computations yield

$$\|\Pi(\mathbf{u}_h \cdot \mathbf{n}) - \mathbf{u}_h \cdot \mathbf{n}\|_{0,F}^2 = \int_F (\mathbf{u}_h \cdot \mathbf{n})^2 - \frac{1}{|F|} \left(\int_F \mathbf{u}_h \cdot \mathbf{n} \right)^2.$$

Let $K \in \mathcal{T}_h^F$ be the hexahedron such that F is one of its faces. Let $\mathbf{F}_K : \hat{K} \rightarrow K$ be chosen such that $F = \mathbf{F}_K(\hat{F})$ with $\hat{F} := [0, 1]^2 \times \{0\}$. Let $\mathbf{F}_F := \mathbf{F}_K|_{\hat{F}} : \hat{F} \rightarrow F$; then $\mathbf{F}_F$ is a bilinear one-to-one map onto F .

Let $\hat{S} \subset \hat{F}$ and $S := \mathbf{F}_F(\hat{S})$. Then

$$\int_S \mathbf{u}_h \cdot \mathbf{n} = \int_{\hat{S}} (\mathbf{u}_h \cdot \mathbf{n}) \circ \mathbf{F}_F J_F, \quad \text{with } J_F := \left| \frac{\partial \mathbf{F}_F}{\partial \hat{x}_1} \times \frac{\partial \mathbf{F}_F}{\partial \hat{x}_2} \right| \quad \text{on } \hat{F}.$$

Therefore, because of (3.9), we have that

$$(\mathbf{u}_h \cdot \mathbf{n}) \circ \mathbf{F}_F = \frac{1}{J_F} \hat{\mathbf{u}}_h \cdot \mathbf{n}_{\hat{F}} \quad \text{on } \hat{F}.$$

Hence, using that $\hat{\mathbf{u}}_h \cdot \mathbf{n}_{\hat{F}}$ is constant on $\hat{F}$ and (3.9) again, we obtain

$$\int_F (\mathbf{u}_h \cdot \mathbf{n})^2 = \int_{\hat{F}} \frac{1}{J_F} (\hat{\mathbf{u}}_h \cdot \mathbf{n}_{\hat{F}})^2 = \left(\int_{\hat{F}} \hat{\mathbf{u}}_h \cdot \mathbf{n}_{\hat{F}} \right)^2 \int_{\hat{F}} \frac{1}{J_F} = \left(\int_F \mathbf{u}_h \cdot \mathbf{n} \right)^2 \int_{\hat{F}} \frac{1}{J_F}.$$

Consequently,

$$\|\Pi(\mathbf{u}_h \cdot \mathbf{n}) - \mathbf{u}_h \cdot \mathbf{n}\|_{0,F}^2 = \left(\int_F \mathbf{u}_h \cdot \mathbf{n} \right)^2 \int_{\hat{F}} \left(\frac{1}{J_F} - \frac{1}{|F|} \right). \quad (4.6)$$

On the other hand, $\exists C_i > 0, i = 1, 2$, such that

$$C_1 h_K^2 \leq J_F(\hat{\mathbf{x}}) \leq C_2 h_K^2 \quad \forall \hat{\mathbf{x}} \in \hat{F}.$$

Indeed, the first inequality is a consequence of the relation

$$J_K = \frac{\partial \mathbf{F}_K}{\partial \hat{x}_1} \times \frac{\partial \mathbf{F}_K}{\partial \hat{x}_2} \cdot \frac{\partial \mathbf{F}_K}{\partial \hat{x}_3} = \frac{\partial \mathbf{F}_F}{\partial \hat{x}_1} \times \frac{\partial \mathbf{F}_F}{\partial \hat{x}_2} \cdot \frac{\partial \mathbf{F}_K}{\partial \hat{x}_3} \quad \text{on } \hat{F},$$

(3.1), (3.2), and (3.12), while the definition of J_F , (3.1), and (3.12) lead to the second one.

Next, we use the notation introduced in (3.4)–(3.5) to write $\mathbf{F}_F(\hat{\mathbf{x}}) = \hat{x}_1 \boldsymbol{\alpha}_1 + \hat{x}_2 \boldsymbol{\alpha}_2 + \hat{x}_1 \hat{x}_2 \boldsymbol{\varepsilon}_3$ and, hence,

$$J_F(\hat{\mathbf{x}}) = |(\boldsymbol{\alpha}_1 + \hat{x}_2 \boldsymbol{\varepsilon}_3) \times (\boldsymbol{\alpha}_2 + \hat{x}_1 \boldsymbol{\varepsilon}_3)| = |(\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2) + \hat{x}_1(\boldsymbol{\alpha}_1 \times \boldsymbol{\varepsilon}_3) + \hat{x}_2(\boldsymbol{\varepsilon}_3 \times \boldsymbol{\alpha}_2)|.$$

Since Γ_1 is a polyhedral surface, F is plane. Then $\boldsymbol{\varepsilon}_3$ lies on the plane determined by $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ (see Fig. 5) and, hence, the three vectors $(\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2)$, $(\boldsymbol{\alpha}_1 \times \boldsymbol{\varepsilon}_3)$, and $(\boldsymbol{\varepsilon}_3 \times \boldsymbol{\alpha}_2)$ have the same direction orthogonal to F ; namely, $\exists a, b, c \in \mathbb{R}$ such that

$$\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2 = a\mathbf{n}, \quad \boldsymbol{\alpha}_1 \times \boldsymbol{\varepsilon}_3 = b\mathbf{n}, \quad \text{and} \quad \boldsymbol{\varepsilon}_3 \times \boldsymbol{\alpha}_2 = c\mathbf{n}.$$

Moreover, since J_F is a non-vanishing continuous function on $\hat{F}$,

$$J_F(\hat{\mathbf{x}}) = \pm (a + b\hat{x}_1 + c\hat{x}_2), \quad \hat{\mathbf{x}} \in \hat{F};$$

the sign depends on the relative orientations of $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, and $\mathbf{n}$, but is the same for all $\hat{\mathbf{x}} \in \hat{F}$. For the sake of definiteness we assume that $J_F(\hat{\mathbf{x}}) = a + b\hat{x}_1 + c\hat{x}_2 \quad \forall \hat{\mathbf{x}} \in \hat{F}$, but the rest of the proof is equally valid in the other case.

FIG. 5: Image of the bilinear map $\mathbf{F}_F$.

Since J_F is an affine function on $\hat{F}$,

$$|F| = \int_{\hat{F}} J_F = J_F(\hat{\mathbf{x}}_0).$$

where $\hat{\mathbf{x}}_0 := (\frac{1}{2}, \frac{1}{2}, 0)$ is the mid-point of $\hat{F}$. On the other hand, using Taylor's formula centered at $\hat{\mathbf{x}}_0$, we have

$$\begin{aligned} \frac{1}{J_F(\hat{\mathbf{x}})} &= \frac{1}{J_F(\hat{\mathbf{x}}_0)} - \frac{1}{J_F^2(\hat{\mathbf{x}}_0)} \left[b \left(\hat{x}_1 - \frac{1}{2} \right) + c \left(\hat{x}_2 - \frac{1}{2} \right) \right] \\ &\quad + \frac{1}{J_F^3(\hat{\mathbf{x}}_0)} \left[b^2 \left(\hat{x}_1 - \frac{1}{2} \right)^2 + 2bc \left(\hat{x}_1 - \frac{1}{2} \right) \left(\hat{x}_2 - \frac{1}{2} \right) + c^2 \left(\hat{x}_2 - \frac{1}{2} \right)^2 \right], \end{aligned}$$

with $\hat{\boldsymbol{\xi}} \in \hat{F}$. The integral on $\hat{F}$ of the second term in the right-hand side above vanishes, whereas $|b| = |\boldsymbol{\alpha}_1 \times \boldsymbol{\varepsilon}_3| \leq \sigma_4 h_K^3$ and $|c| = |\boldsymbol{\varepsilon}_3 \times \boldsymbol{\alpha}_2| \leq \sigma_4 h_K^3$ because of (3.14). Then

$$\int_{\hat{F}} \left(\frac{1}{J_F} - \frac{1}{|F|} \right) \leq \frac{\sigma_4^2}{6C_1^3}.$$

Finally, using (3.15) and the Cauchy-Schwarz inequality, we obtain

$$\left(\int_F \mathbf{u}_h \cdot \mathbf{n} \right)^2 = \left(\int_F \mathbf{w}_h \cdot \mathbf{n} \right)^2 \leq |F| \int_F (\mathbf{w}_h \cdot \mathbf{n})^2 \leq C_2 h_K^2 \int_F (\mathbf{w}_h \cdot \mathbf{n})^2.$$

Thus, by using the last two inequalities to bound (4.6) and summing up for all $F \in \mathcal{T}_h^1$, we conclude the proof. $\square$

The following lemma is the basic tool to prove double-order error estimates for the approximation of the eigenvalues:

Lemma 4.10 *For all $(\mathbf{f}, \mathbf{g})$ and $(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) \in \mathcal{G}$,*

$$|b((T - T_h)(\mathbf{f}, \mathbf{g}), (\tilde{\mathbf{f}}, \tilde{\mathbf{g}}))| \leq Ch^{2t} \|(\mathbf{f}, \mathbf{g})\|_{\mathcal{X}} \|(\tilde{\mathbf{f}}, \tilde{\mathbf{g}})\|_{\mathcal{X}}.$$

Proof. Let $(\mathbf{f}, \mathbf{g})$ and $(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) \in \mathcal{G}$. We denote

$$\begin{aligned} (\mathbf{u}, \mathbf{w}) &:= T(\mathbf{f}, \mathbf{g}), & (\mathbf{u}_h, \mathbf{w}_h) &:= T_h(\mathbf{f}, \mathbf{g}), \\ (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) &:= T(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}), & (\tilde{\mathbf{u}}_h, \tilde{\mathbf{w}}_h) &:= T_h(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}). \end{aligned}$$

Consider the Helmholtz-like decompositions of $\mathbf{u}_h$ and $\tilde{\mathbf{u}}_h$ as in Lemma 4.3:

$$\mathbf{u}_h = \nabla \xi + \chi, \quad \tilde{\mathbf{u}}_h = \nabla \tilde{\xi} + \tilde{\chi}.$$

Proceeding as in the proof of Lemma 4.4, we have

$$a((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), (\tilde{\mathbf{u}}_h, \tilde{\mathbf{w}}_h)) = - \int_{\Omega_F} \rho_F \mathbf{f} \cdot \tilde{\chi},$$

whereas a similar argument leads to

$$a((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), (\tilde{\mathbf{u}}, \tilde{\mathbf{w}})) = b((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), (\tilde{\mathbf{f}}, \tilde{\mathbf{g}})) + \int_{\Omega_F} \rho_F \tilde{\mathbf{f}} \cdot \chi.$$

Thus, subtracting these two equations we obtain

$$\begin{aligned} b((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), (\tilde{\mathbf{f}}, \tilde{\mathbf{g}})) &= a((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), (\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h, \tilde{\mathbf{w}} - \tilde{\mathbf{w}}_h)) \\ &\quad - \int_{\Omega_F} \rho_F \tilde{\mathbf{f}} \cdot \chi - \int_{\Omega_F} \rho_F \mathbf{f} \cdot \tilde{\chi}. \end{aligned}$$

The continuity of a and Lemma 4.6 yields the following bound for the first term on the right-hand side above:

$$a((\mathbf{u} - \mathbf{u}_h, \mathbf{w} - \mathbf{w}_h), (\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h, \tilde{\mathbf{w}} - \tilde{\mathbf{w}}_h)) \leq Ch^{2t} \|(\mathbf{f}, \mathbf{g})\|_{\mathcal{X}} \|(\tilde{\mathbf{f}}, \tilde{\mathbf{g}})\|_{\mathcal{X}}.$$

The remaining two integral terms are similar, so we only consider one of them. For the first one, since $(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) \in \mathcal{G}$, there exist $\tilde{\varphi} \in H^1(\Omega_F)$ such that $\tilde{\mathbf{f}} = \nabla \tilde{\varphi}$. Applying the Gauss theorem and using that $\operatorname{div} \chi = 0$ and $(\nabla \xi, \mathbf{w}_h) \in \mathcal{G} \subset \mathcal{V}$, we obtain

$$\int_{\Omega_F} \rho_F \tilde{\mathbf{f}} \cdot \chi = \int_{\Gamma_1} \rho_F \tilde{\varphi} \chi \cdot \mathbf{n} = \int_{\Gamma_1} \rho_F \tilde{\varphi} (\mathbf{u}_h \cdot \mathbf{n} - \nabla \xi \cdot \mathbf{n}) = \int_{\Gamma_1} \rho_F \tilde{\varphi} (\mathbf{u}_h \cdot \mathbf{n} - \mathbf{w}_h \cdot \mathbf{n}).$$

Since $(\mathbf{u}_h, \mathbf{w}_h) \in \mathcal{V}_h$, because of (3.15) we have that $\Pi(\mathbf{u}_h \cdot \mathbf{n}) = \Pi(\mathbf{w}_h \cdot \mathbf{n})$. Hence,

$$\begin{aligned} \int_{\Omega_F} \rho_F \tilde{\mathbf{f}} \cdot \boldsymbol{\chi} &= \int_{\Gamma_1} \rho_F \tilde{\varphi} [\mathbf{u}_h \cdot \mathbf{n} - \Pi(\mathbf{u}_h \cdot \mathbf{n}) + \Pi(\mathbf{w}_h \cdot \mathbf{n}) - \mathbf{w}_h \cdot \mathbf{n}] \\ &= \int_{\Gamma_1} \rho_F [\tilde{\varphi} - \Pi\tilde{\varphi}] [\mathbf{u}_h \cdot \mathbf{n} - \Pi(\mathbf{u}_h \cdot \mathbf{n}) + \Pi(\mathbf{w}_h \cdot \mathbf{n}) - \mathbf{w}_h \cdot \mathbf{n}] \\ &\leq \rho_F \|\tilde{\varphi} - \Pi\tilde{\varphi}\|_{0,\Gamma_1} \left[\|\mathbf{u}_h \cdot \mathbf{n} - \Pi(\mathbf{u}_h \cdot \mathbf{n})\|_{0,\Gamma_1} + \|\Pi(\mathbf{w}_h \cdot \mathbf{n}) - \mathbf{w}_h \cdot \mathbf{n}\|_{0,\Gamma_1} \right]. \end{aligned}$$

Now, since $r > \frac{1}{2}$, by using the standard error estimate for the L^2 projection onto piecewise constants, the trace theorem, and (2.9), we obtain

$$\|\tilde{\varphi} - \Pi\tilde{\varphi}\|_{0,\Gamma_1} \leq Ch \|\tilde{\varphi}\|_{1,\Gamma_1} \leq Ch \|\nabla \tilde{\varphi}\|_{r,\Omega_F} = Ch \|\tilde{\mathbf{f}}\|_{r,\Omega_F} \leq Ch \|(\tilde{\mathbf{f}}, \tilde{\mathbf{g}})\|_{\mathcal{X}}.$$

On the other hand, since $\|\Pi\psi\|_{0,\Gamma_1} \leq \|\psi\|_{0,\Gamma_1} \quad \forall \psi \in L^2(\Gamma_1)$, we have

$$\begin{aligned} &\|\Pi(\mathbf{w}_h \cdot \mathbf{n}) - \mathbf{w}_h \cdot \mathbf{n}\|_{0,\Gamma_1} \\ &\leq \|\Pi(\mathbf{w}_h \cdot \mathbf{n} - \mathbf{w} \cdot \mathbf{n})\|_{0,\Gamma_1} + \|\Pi(\mathbf{w} \cdot \mathbf{n}) - \mathbf{w} \cdot \mathbf{n}\|_{0,\Gamma_1} + \|\mathbf{w} \cdot \mathbf{n} - \mathbf{w}_h \cdot \mathbf{n}\|_{0,\Gamma_1} \\ &\leq Ch^{\min\{1, \frac{1}{2}+s\}} \|\mathbf{w} \cdot \mathbf{n}\|_{\frac{1}{2}+s,\Gamma_1} + 2 \|\mathbf{w} - \mathbf{w}_h\|_{1,\Omega_S} \\ &\leq Ch^t \|(\mathbf{f}, \mathbf{g})\|_{\mathcal{X}}, \end{aligned}$$

where we have used the error estimate for the L^2 projection again, the trace theorem, Lemma 4.6, and Theorem 2.4.

Finally, from Lemma 4.9, the trace theorem, and the boundedness of T_h , we obtain

$$\|\Pi(\mathbf{u}_h \cdot \mathbf{n}) - \mathbf{u}_h \cdot \mathbf{n}\|_{0,\Gamma_1} \leq Ch \|(\mathbf{f}, \mathbf{g})\|_{\mathcal{X}}.$$

Therefore, the last four inequalities yield

$$\int_{\Omega_F} \rho_F \tilde{\mathbf{f}} \cdot \boldsymbol{\chi} \leq Ch^{1+t} \|(\mathbf{f}, \mathbf{g})\|_{\mathcal{X}} \|(\tilde{\mathbf{f}}, \tilde{\mathbf{g}})\|_{\mathcal{X}},$$

which, given that $t \leq 1$, allows us to conclude the proof. $\square$

Now we are in a position to prove a double-order error estimate for the approximation of the eigenvalues. Let $\lambda = 1/\mu$ (with μ being an eigenvalue of T) be a positive eigenvalue of Problem 2.2 with algebraic multiplicity m and eigenspace $\mathcal{E}$. Let $\lambda_{hi} = 1/\mu_{hi}$, $i = 1, \dots, m$, (with μ_{hi} being eigenvalues of T_h) be the eigenvalues of Problem 4.1 converging to λ , and let $\mathcal{E}_h$ be the direct sum of their corresponding associated eigenspaces. The following result holds true:

Theorem 4.11 *There exists a strictly positive constant C such that*

$$|\lambda - \lambda_{hi}| \leq Ch^{2t}, \quad i = 1, \dots, m.$$

Proof. Since T and T_h are self-adjoint with respect to b , by specializing the results of Theorem 7.3 from Babuška & Osborn (1991) to our case, for $i = 1, \dots, m$ we have

$$|\mu - \mu_{hi}| \leq C \left[\sup_{(\mathbf{0}, \mathbf{0}) \neq (\mathbf{f}, \mathbf{g}), (\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) \in \mathcal{E}} \frac{|b((T - T_h)(\mathbf{f}, \mathbf{g}), (\tilde{\mathbf{f}}, \tilde{\mathbf{g}}))|}{\|(\mathbf{f}, \mathbf{g})\|_{\mathcal{H}} \|(\tilde{\mathbf{f}}, \tilde{\mathbf{g}})\|_{\mathcal{H}}} + \|(T - T_h)|_{\mathcal{E}}\|_{\mathcal{H}}^2 \right],$$

where we have also used that the inner product b defines a norm on $\mathcal{H}$ equivalent to $\|\cdot\|_{\mathcal{H}}$.

Since $T : \mathcal{H} \rightarrow \mathcal{X}$ is a bounded operator, for $(\mathbf{f}, \mathbf{g})$ and $(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) \in \mathcal{E}$, there holds

$$\|(\mathbf{f}, \mathbf{g})\|_{\mathcal{X}} \leq \frac{C}{\mu} \|(\mathbf{f}, \mathbf{g})\|_{\mathcal{H}} \quad \text{and} \quad \|(\tilde{\mathbf{f}}, \tilde{\mathbf{g}})\|_{\mathcal{X}} \leq \frac{C}{\mu} \|(\tilde{\mathbf{f}}, \tilde{\mathbf{g}})\|_{\mathcal{H}}.$$

Hence, by virtue of Lemmas 4.10 and 4.6, and the continuity of the inclusion $\mathcal{X} \hookrightarrow \mathcal{H}$, we have

$$|\mu - \mu_{hi}| \leq Ch^{2t}, \quad i = 1, \dots, m.$$

Therefore, the theorem is a consequence of the above inequality and the fact that $\lambda = 1/\mu$ and $\lambda_{hi} = 1/\mu_{hi}$, $i = 1, \dots, m$. $\square$

5 A numerical test

The main goal of this section is to study the need for the hypothesis on $\{\mathcal{T}_h^F\}$ being a family of asymptotically parallelepiped meshes. We refer to Bermúdez *et al.* (2001a, 2001b) for numerical experiments confirming the theoretical results proved in this paper, in particular regarding the ability of the method to deal with nonmatching fluid and solid grids. The performance of the method has been also shown in these references to be significantly advantageous as compared with similar discretizations based on tetrahedral finite elements. Further computational results can be found in Gamallo (2002).

For the sake of simplicity we consider a purely acoustic 2D problem: a cavity, with perfectly rigid walls, filled with fluid. The domain is the unit square $\Omega_F := [0, 1]^2$ and the physical parameters of the fluid, ρ_F and c , are set equal to 1. The vibration frequencies of this problem are analytically known:

$$\omega_{mn} = \pi \sqrt{m^2 + n^2} \quad m, n = 0, 1, 2, \dots$$

To test the method with a family of non asymptotically parallelogram meshes, we have considered partitions of the domain into $N \times N$ congruent trapezoids, all similar to the trapezoid with vertices $(0, 0)$, $(\frac{1}{2}, 0)$, $(\frac{1}{2}, \frac{2}{3})$ and $(0, \frac{1}{3})$, as shown in Fig. 6. Clearly the elements of these meshes are uniformly shape-regular but far from being parallelograms. They have been used by Arnold *et al.* (2001, 2002a, 2002b) to show that the order of convergence of some quadrilateral finite elements deteriorate, even on shape-regular meshes.

We have solved the problem with uniform meshes of $N \times N$ squares and with a family of asymptotically parallelogram but not uniform meshes, as well. The latter consist of uniform refinements of an initial non-uniform mesh, as shown in Fig. 7. It is simple to prove that this family satisfies assumption (3.14). Let us remark that this kind of meshes arise naturally when the domain is not a parallelogram or a union of ‘disjoint’ parallelograms.

FIG. 6: Non asymptotically parallelogram trapezoid meshes in $\Omega_F = [0, 1]^2$.

FIG. 7: Asymptotically parallelogram non-uniform meshes in $\Omega_F = [0, 1]^2$.

Tables 1, 2, and 3 show the five smallest vibration frequencies ω_{mn} computed with each family of meshes. Each table includes the values of the vibration frequencies obtained by extrapolating the computed ones (that we denote $\hat{\omega}_{mn}$, $\tilde{\omega}_{mn}$, and $\tilde{\omega}_{mn}$, respectively), the computed orders of convergence, and the exact values of the vibration frequencies.

TABLE 1: Vibration frequencies computed on square meshes.

	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$\hat{\omega}_{mn}$	Order	Exact
ω_{10}	3.161816	3.146642	3.142855	3.141908	3.141590	2.00	3.141593
ω_{01}	3.161816	3.146642	3.142855	3.141908	3.141590	2.00	3.141593
ω_{11}	4.471483	4.450023	4.444667	4.443329	4.442878	2.00	4.442883
ω_{20}	6.445663	6.323632	6.293283	6.285709	6.283265	2.01	6.283185
ω_{02}	6.445663	6.323632	6.293283	6.285709	6.283265	2.01	6.283185

TABLE 2: Vibration frequencies computed on asymptotically parallelogram non-uniform meshes.

	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$\tilde{\omega}_{mn}$	Order	Exact
ω_{10}	3.162382	3.146826	3.142903	3.141920	3.141586	1.99	3.141593
ω_{01}	3.169306	3.148542	3.143331	3.142027	3.141588	2.00	3.141593
ω_{11}	4.482441	4.452946	4.445409	4.443515	4.442850	1.97	4.442883
ω_{20}	6.425867	6.319368	6.292258	6.285455	6.283086	1.98	6.283185
ω_{02}	6.459253	6.327803	6.294371	6.285984	6.283066	1.98	6.283185

TABLE 3: *Vibration frequencies computed on non asymptotically parallelogram trapezoid meshes.*

	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$\tilde{\omega}_{mn}$	Order	Exact
ω_{10}	3.216744	3.206698	3.204210	3.203576	3.203375	2.01	3.141593
ω_{01}	3.223999	3.208523	3.204667	3.203690	3.203373	2.00	3.141593
ω_{11}	4.553755	4.536108	4.531724	4.530611	4.530253	2.00	4.442883
ω_{20}	6.515469	6.433513	6.413410	6.408396	6.406806	2.02	6.283185
ω_{02}	6.572428	6.447991	6.417058	6.409310	6.406764	2.01	6.283185

Notice that the method converges with almost perfect quadratic order in all cases. However, for the non asymptotically parallelogram meshes, it converges to wrong vibration frequencies! Instead, the results obtained with the asymptotically parallelogram non-uniform meshes are essentially of the same quality as those obtained with the square meshes.

On the other hand, we note that there exists an almost constant ratio between the extrapolated values obtained with the non asymptotically parallelogram and the square meshes:

$$\frac{\tilde{\omega}_{mn}}{\hat{\omega}_{mn}} \approx 1.01966;$$

namely, the vibration frequencies extrapolated from the values computed with these non asymptotically parallelogram trapezoid meshes have a systematic error of around 2%. Table 4 shows the corresponding ratios for different refinement levels.

TABLE 4: *Ratio between vibration frequencies computed with non asymptotically parallelogram and square meshes, $\tilde{\omega}_{mn}/\hat{\omega}_{mn}$.*

	$N = 8$	$N = 16$	$N = 32$	$N = 64$
ω_{10}	1.017372	1.019086	1.019522	1.019627
ω_{01}	1.019667	1.019666	1.019664	1.019663
ω_{11}	1.018399	1.019345	1.019587	1.019643
ω_{20}	1.010830	1.017376	1.019088	1.019518
ω_{02}	1.019667	1.019666	1.019668	1.019664

It can be seen from this table that the ratios converge to $\tilde{\omega}_{mn}/\hat{\omega}_{mn} \approx 1.01966$ as the meshes get finer.

This apparently strange behavior can be explained if one compares the element matrices in the two cases. Indeed, according to the notation of Fig. 8, if ϕ_i^γ and ϕ_i^0 , $i = 1, \dots, 4$, denote the Raviart-Thomas basis functions in elements K_γ and K_0 , respectively, explicit computations allow us to show that

$$\int_{K_\gamma} \operatorname{div} \phi_i^\gamma \operatorname{div} \phi_j^\gamma = C_\gamma \int_{K_0} \operatorname{div} \phi_i^0 \operatorname{div} \phi_j^0 \quad \forall i, j = 1, \dots, 4,$$

where

$$C_\gamma := \frac{1}{2 \tan \gamma} \log \left(\frac{1 + \tan \gamma}{1 - \tan \gamma} \right),$$

FIG. 8: Elements: trapezoid K_γ and square K_0 .

with γ being the angle shown in Fig. 8. Thus, the stiffness matrix on the $N \times N$ trapezoid mesh is exactly C_γ times the corresponding matrix on the $N \times N$ square mesh.

For the trapezoids used above, $\tan \gamma = \frac{1}{3}$ and then

$$C_\gamma = 1.03972077... \approx \left(\frac{\tilde{\omega}_{mn}}{\hat{\omega}_{mn}} \right)^2.$$

On the other hand, the mass matrices computed on the trapezoid meshes can be thought of as mass matrices on the square meshes computed by means of a peculiar integration rule. This is the reason why the ratio between the vibration frequencies (i.e., the square root of the eigenvalues) converge to $\sqrt{C_\gamma}$ as the meshes get finer.

Finally, the pressure field for the first three vibration modes computed on the trapezoid meshes with refinement levels $N = 16$ and $N = 32$ are depicted in Figs. 9 to 11.

FIG. 9: Pressure field for ω_{10} . Trapezoid meshes: $N = 16$ (left) and $N = 32$ (right).

‘Undesired’ non physical oscillations clearly related with the shape of the elements appear in the second and third vibration modes, although not in the first one. It is apparent that the reason for this different behavior is the relation between the anisotropy of the discrete eigenfunctions and the orientation of the meshes.

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FIG. 10: *Pressure field for ω_{01} . Trapezoid meshes: $N = 16$ (left) and $N = 32$ (right).*

FIG. 11: *Pressure field for ω_{11} . Trapezoid meshes: $N = 16$ (left) and $N = 32$ (right).*

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REFERENCES

- ALONSO, A. & VALLI, A. (1999) An optimal domain decomposition preconditioner for low-frequency time-harmonic Maxwell equations. *Math. Comp.*, **68**, 607–631.
- ARNOLD, D. N., BOFFI, D. & FALK, R. S. (2002a) Approximation by quadrilateral finite elements. *Math. Comp.*, **71**, 909–922.
- ARNOLD, D. N., BOFFI, D. & FALK, R. S. (2002b) Quadrilateral $H(\text{div})$ finite elements. *SIAM J. Numer. Anal.* (to appear).
- ARNOLD, D. N., BOFFI, D., FALK, R. S. & GASTALDI, L. (2001) Finite element approximation on quadrilateral meshes. *Comm. Numer. Methods Engrg.*, **17**, 805–812.
- BABUŠKA, I. & OSBORN, J. (1991) Eigenvalue Problems. *Handbook of Numerical Analysis, Vol. II.* (Ciarlet, P. G. & Lions, J. L., eds.) Amsterdam: North-Holland, pp. 641–787.

- BATHE, K. J., NITIKITPAIBOON, C. & WANG, X. (1995) A mixed displacement-based finite element formulation for acoustic fluid-structure interaction. *Comput. & Structures*, **56**, 225–237.
- BERMÚDEZ, A., DURÁN, R., MUSCHIETTI, M. A., RODRÍGUEZ, R. & SOLOMIN, J. (1995) Finite element vibration analysis of fluid-solid systems without spurious modes. *SIAM J. Numer. Anal.*, **32**, 1280–1295.
- BERMÚDEZ, A., GAMALLO, P. & RODRÍGUEZ, R. (2001a) A hexahedral face element for elastoacoustic vibration problems. *J. Acoustical Soc. Amer.*, **109**, 422–425.
- BERMÚDEZ, A., GAMALLO, P. & RODRÍGUEZ, R. (2001b) An hexahedral face element method for the displacement formulation of structural acoustics problems. *J. Comput. Acoust.*, **9**, 911–918.
- BERMÚDEZ, A., HERVELLA-NIETO, L. & RODRÍGUEZ, R. (1999) Finite element computation of three dimensional elastoacoustic vibrations. *J. Sound & Vibration*, **219**, 277–304.
- BERMÚDEZ, A. & RODRÍGUEZ, R. (1994) Finite element computation of the vibration modes of a fluid-solid system. *Comput. Methods Appl. Mech. Engrg.*, **119**, 355–370.
- BREZZI, F. & FORTIN, M. (1991) *Mixed and Hybrid Finite Element Methods*. New York: Springer.
- CIARLET, P. G. (1991) Basic error estimates for elliptic problems. *Handbook of Numerical Analysis, Vol. II* (Ciarlet, P. G. & Lions, J. L., eds.) Amsterdam: North-Holland, pp. 17–351.
- CLÉMENT, PH. (1975) Approximation by finite element functions using local regularization. *RAIRO, Sér. Rouge Anal. Numér.*, **9**, 77–84.
- CONCA, C., PLANCHARD, J. & VANNINATHAN, M. (1995) *Fluids and Periodic Structures*. Paris: Masson.
- DAUGE, M. (1988) Elliptic boundary value problems on corner domains: smoothness and asymptotics of solutions. *Lecture Notes in Mathematics*, **1341**, Berlin: Springer-Verlag.
- DUPONT, T. & SCOTT, R. (1980) Polynomial approximation of functions in Sobolev spaces *Math. Comp.*, **34**, 441–463.
- DURÁN, R., HERVELLA-NIETO, L., LIBERMAN, E., RODRÍGUEZ, R. & SOLOMIN, J. (2000) Finite element analysis of the vibration problem of a plate coupled with a fluid. *Numer. Math.*, **86**, 591–616.
- EVERSTINE, G. C. (1981) A symmetric potential formulation for fluid-structure interaction. *J. Sound & Vibration*, **79**, 157–160.
- GAMALLO, P (2002) *Contribución al estudio matemático de problemas de simulación elastoacústica y control activo del ruido*. Tesis de Doctor en Matemáticas, Universidad de Santiago de Compostela, Spain, 2002.
- GASTALDI, L. (1996) Mixed finite element methods in fluid structure systems. *Numer. Math.*, **74**, 153–176.
- GIRAULT, V. & RAVIART, P. A. (1986) *Finite Element Methods for Navier-Stokes Equations*. Berlin: Springer-Verlag.
- HAMD, M., OUSSET, Y. & VERCHERY, G. (1978) A displacement method for the analysis of vibrations of coupled fluid-structure systems. *Internat. J. Numer. Methods Engrg.*, **13**, 139–150.
- KATO, T. (1976) *Perturbation Theory for Linear Operators*, Berlin: Springer.
- KIEFLING, L. & FENG, G. C. (1976) Fluid-structure finite element vibration analysis. *AIAA J.*, **14**, 199–203.

- MORAND, H. J-P. & OHAYON, R. (1995) *Fluid Structure Interaction*. Chichester: J. Wiley & Sons.
- MORAND, H. & OHAYON, R. (1979) Substructure variational analysis of the vibrations of coupled fluid-structure systems. Finite element results. *Internat. J. Numer. Methods Engrg.*, **14**, 741–755.
- NÉDÉLEC, J. C. (1980) Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, **35**, 315–341.
- OHAYON, R. & SOIZE, C. (1998) *Structural Acoustics and Vibration*, New York: Academic Press.
- OLSON, L. G. & BATHE, K. J. (1983) A study of displacement-based fluid finite elements for calculating frequencies of fluid and fluid-structure systems. *Nuclear Engrg. & Design*, **76**, 137–151.
- RAVIART, P. A. & THOMAS, J. M. (1977) A mixed finite element method for second order elliptic problems. *Mathematical Aspects of Finite Element Methods*. (Galligani, I. & Magenes, E., eds.) *Lecture Notes in Mathematics*, **606**, Berlin: Springer-Verlag, pp. 292–315.
- RODRÍGUEZ, R. & SOLOMIN, J. (1996) The order of convergence of eigenfrequencies in finite element approximations of fluid-structure interaction problems. *Math. Comp.*, **65**, 1463–1475.
- THOMAS, J. M. (1977) *Sur l'Analyse Numérique des Méthodes d'Éléments Finis Hybrides et Mixtes*. Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris.
- WANG, X. & BATHE, K. J. (1997) Displacement/pressure based mixed finite element formulations for acoustic fluid-structure interaction problems. *Internat. J. Numer. Methods Engrg.*, **40**, 2001–2017.
- ZIENKIEWICZ, O. C. & TAYLOR, R. L. (1991) *The Finite Element Method*. Vol. 2. London: McGraw-Hill.