Existence, Uniqueness, and Stability of Generalized Solutions of an Initial-Boundary Value Problem for a Degenerating Quasilinear Parabolic Equation

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1. INTRODUCTION

In [3] we presented a spatially one-dimensional mathematical model for the settling and consolidation of a flocculated suspension under the influence of gravity. This model can be formulated as an initial-boundary value problem of a scalar quasilinear partial differential equation of second order parabolic type for the local volumetric solid concentration as a function of height and time. This equation degenerates into first order hyperbolic type if the concentration is less than a critical value, i.e., on an interval of solution values. For this reason, previous results by Gilding [6] and Zhao [18], in which the existence of continuous solutions was shown under the assumption of degeneracy only for singular solution values, cannot be applied here, and discontinuous solutions have to be considered in a more general class. An appropriate definition of generalized solutions to the initial-boundary value problem was formulated in [3], from which entropy boundary and jump conditions were derived subsequently.

In this paper, we prove the existence, stability, and uniqueness of generalized solutions of such initial-boundary value problems. In Section 2, we recall some notation and definitions from [3]. The existence of generalized solutions is shown here by the vanishing viscosity method. Therefore we first consider the corresponding family of regularized parabolic initial-boundary value problems with a positive viscosity parameter \( \varepsilon > 0 \). The

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existence of a unique smooth solution for every fixed value of $\varepsilon$ is shown in Section 3. In Section 4, we show that the family of corresponding smooth solutions possesses a limit for $\varepsilon \to 0$, and that this limit is a generalized solution. In Section 5, we prove the stability of generalized solutions, from which uniqueness follows immediately.

2. THE INITIAL-BOUNDARY VALUE PROBLEM

We study the following initial-boundary value problem [3]:

$$\frac{\partial u}{\partial t} + a(u) \frac{\partial u}{\partial x} = f(u(t,x)) \text{ on } \Omega_T = (0,1) \times (0,T), \quad (2.1)$$

$$\left. u(x,0) = u_0(x) \right|_{x=0} \text{ for } 0 \leq x \leq 1, \quad (2.2)$$

$$\left. a(u) \frac{\partial u}{\partial x} \right|_{x=0} = 0 \text{ for } 0 < t \leq T, \quad (2.3)$$

$$u(1,t) = \bar{u}_0(t) \text{ for } 0 < t \leq T. \quad (2.4)$$

Here, $u$ is the scalar function to be determined on $Q_T$ and $f$ is the flux density function consisting of a linear convection part $q(t)u$ and a nonlinear part $f_{bh}(u)$, i.e., $f(u(t,x)) = q(t)u + f_{bh}(u)$, where $q$ is a continuously differentiable nonpositive function defined on $[0,T]$ and $f_{bh}$ is a twice continuously differentiable function satisfying

$$f_{bh}(0) = f_{bh}(1) = 0, \quad f_{bh}(u) \leq 0 \text{ for } 0 \leq u \leq 1, \quad f_{bh}'(1) = 0, \quad (2.5)$$

where we put formally

$$f_{bh}(u) = 0 \text{ for } u < 0 \text{ or } u > 1. \quad (2.6)$$

The diffusion coefficient $a$ on the right-hand side of (2.1) is assumed to be a known continuously differentiable function of $u$ with

$$a(u) = \begin{cases} 
0 & \text{ for } u \leq \phi_c \\
> 0 & \text{ for } \phi_c < u < 1 \\
0 & \text{ for } u \geq 1,
\end{cases} \quad (2.7)$$

where $\phi_c$ is a critical solution value. This means that (2.1) is hyperbolic for $u \leq \phi_c$, parabolic for $\phi_c < u < 1$, and hyperbolic for $u \geq 1$. For $u \geq 1$, Eq. (2.1) degenerates into a linear advection equation. We will see that the generalized solution to the initial-boundary value problem assumes values
from the unit interval \([0, 1]\) almost everywhere, such that it is sufficient to
cosider the degeneracy only for \(0 \leq u \leq \phi_c\) and for \(u = 1\). We require that

\[
0 \leq u_0(x) \leq 1, \quad u_0 \in C^1[0, 1 - \delta] \cap C^2(1 - \delta, 1], \quad \delta > 0, \quad (2.8)
\]

\[
\bar{u}(t) \in C^2[0, T], \quad 0 \leq \bar{u}(t) \leq 1, \quad (2.9)
\]

and that the following \textit{first order compatibility conditions} be satisfied:

\[
u_0(1) = u_1(0), \quad (2.10)
\]

\[
-q(0)u_0'(1) - f_{bh}(u_0(1))u_0'(1)
\]

\[
-\left(a'(u_1(1))(u_0'(1))^2 + a(u_0(1)) + u_0'(1)\right) = \bar{u}(0), \quad (2.11)
\]

\[
a(u_0(0))u_0'(0) - f_{bh}(u_0(0)) = 0. \quad (2.12)
\]

These conditions are necessary for the existence of a smooth solution from
\(C^{2,1}([0, T])\) of the corresponding regularized, parabolic problem. Finally, we
define

\[
P := \{t \in [0, T] \mid a(\bar{u}(t)) > 0\}, \quad H := [0, T] \setminus P.
\]

The initial-boundary value problem (2.1)–(2.4) describes the settling and
consolidation of a flocculated suspension in a one-dimensional ideal contin-
uous thickener of height one: \(u\) denotes the volumetric solid concentra-
tion, \(q(t)\) the volume-averaged velocity of the suspension which can be
controlled externally, \(f_{bh}\) the Kynch batch flux density function \([8, 13]\), and \(\phi_c\) the critical concentration value at which solid flocs begin to touch each
other. In this application, the diffusion coefficient is

\[
a(u) = -\frac{f_{bh}(u)a_c'(u)}{L \Delta \rho g u},
\]

where \(a_c\) is the effective solid stress which is assumed to be constant for
flocs not in touch with each other, i.e., for \(u \leq \phi_c\), we have \(a_c'(\phi) = 0\). Furthermore, \(L\) is the height of the thickener feeding level, \(g\) the
acceleration of gravity, and \(\Delta \rho > 0\) the difference of solid and fluid mass
densities. Condition (2.2) corresponds to prescribing an initial concentra-
tion distribution, condition (2.3) to reducing the solid volume flux at the
discharge surface sink at \(x = 0\) to its convective part \(q(t)u(0, t)\), and
condition (2.4) to continuous feeding of the thickener at level \(x = 1\) with
fresh suspension. See \([3, 4, 13]\) for more details on the sedimentation
model.
2.1. Definition of generalized solutions

As Eq. (2.1) is of nonlinear hyperbolic type for \( u \leq \phi_r \), it is evident that solutions of the initial-boundary value problem (2.1)–(2.4) might be discontinuous despite smooth initial data. Thus we consider generalized discontinuous solutions in the space \( BV(Q_T) \) of all functions that are defined and summable on \( Q_T \) and whose generalized first derivatives are Borel measures, and for which

\[
\int_{Q_T} u \cdot \nabla \varphi \, dx \, dt = -\int_{Q_T} \varphi \cdot \left( \frac{\partial u}{\partial x} (dx \, dt), \frac{\partial u}{\partial t} (dx \, dt) \right)^T \quad \forall \varphi \in C_0^\infty(Q_T),
\]

where \( \nabla \varphi = (\varphi_x, \varphi_t)^T \). We denote by \( \nu = (\nu_x, \nu_t) \) the normal to the set of discontinuities \( \Gamma_\alpha \) and by \( u^\pm \) corresponding approximate limits of \( u \) with respect to \( \pm \nu \). We set for notational convenience \( u_\pm := \min(u^-, u^+) \), \( u^\pm := \max(u^-, u^+) \), and \( \bar{u} := \frac{1}{2}(u^+ + u^-) \). Similarly, we denote by \( u'(x,t) \) and \( u'(x,t) \) the right and left approximate limits of \( u(\cdot,t) \) as a function of \( x \) and set \( \bar{u}(x,t) := \frac{1}{2}(u'(x,t) + u'(x,t)) \). Furthermore, we set \( I(a,b) := [\min(a,b), \max(a,b)] \), and let \( H = H_1 \) be the one-dimensional Hausdorff measure. To keep the chain rule for the differentiation of \( (f \circ u)(x) \) valid in \( BV \), the composition needs to be replaced by the functional superposition

\[
f(u(x,t)) := \int_0^1 f(\tau u^+(x,t) + (1-\tau)u^-(x,t)) \, d\tau.
\]

The definition of generalized solutions is based on an integral inequality with Kružkov entropy functions and corresponding fluxes [7].

**Definition 2.1.** A function \( u \in L^\infty(Q_T) \cap BV(Q_T) \) is a generalized solution of the initial-boundary value problem (2.1) to (2.4) if the following conditions are satisfied:

1. There exists a function \( g \in L^2(Q_T) \) such that for \( \tau = \sqrt{\alpha(t)} \) there holds

\[
\int_{Q_T} \varphi g \, dx \, dt = \int_{Q_T} \varphi \frac{\partial u}{\partial x} \, dx \, dt \quad \forall \varphi \in C_0^\infty(Q_T). \tag{2.13}
\]
The function $u$ satisfies the integral inequality

$$\int_{Q_T} \left[ |u - k| \frac{\partial \varphi}{\partial t} + \text{sgn}(u - k) \left[ f(u, t) - f(k, t) - a(u) \frac{\partial u}{\partial x} \right] \frac{\partial \varphi}{\partial x} \right] \, dx \, dt$$

$$+ \int_0^T \left[ -\text{sgn}(\bar{u}_1(t) - k) \left[ f(\gamma u, t) - f(k, t) + \gamma \left( a(u) \frac{\partial u}{\partial x} \right) \right] \varphi(1, t) \right. + \left. \left[ \text{sgn}(\gamma u - k) - \text{sgn}(\bar{u}_1(t) - k) \right] \left( A(\gamma u) - A(k) \right) \frac{\partial \varphi}{\partial x}(1, t) \right] \, dt$$

$$\geq 0$$  \hspace{1cm} (2.14)

for all $\varphi \in C^\infty((0, 1] \times [0, T])$ where $\varphi \geq 0$ and $\text{supp} \varphi \subset (0, 1] \times (0, T)$, for all $k \in \mathbb{R}$ and $A(u) = \int_0^u a(\tau) \, d\tau$.

3. For almost all $t \in [0, T]$,

$$\gamma \left( -a(u) \frac{\partial u}{\partial x} + f_{bk}(u) \right) \bigg|_{t=0} = 0.$$  \hspace{1cm} (2.15)

4. For almost all $x \in [0, 1]$,

$$(\gamma u)(0, x) = u_0(x).$$  \hspace{1cm} (2.16)

In this definition we use

**Lemma 2.1** [16]. If $\partial v / \partial t$ and $\partial v / \partial x$ are measures (of bounded variation, which will not be mentioned anymore) for $v(x, t) \in L^1(Q_T)$, then for almost all $x \in [0, 1]$ and f.a.a. $t \in [0, T]$, the traces $(\gamma v)(x, 0) \equiv \lim_{t \to 0^+} \bar{v}(x, t)$, $(\gamma v)(0, t) \equiv \lim_{x \to 0^+} \bar{v}(x, t)$, and $(\gamma v)(1, t) \equiv \lim_{x \to 1^-} \bar{v}(x, t)$ exist, and $(\gamma v)(x, 0) \in L^1(0, 1)$, $(\gamma v)(0, t), (\gamma v)(1, t) \in L^1(0, T)$, where $\bar{v}(x, t)$ is a function equivalent to $v(x, t)$ on $Q_T$. The traces $\gamma v$ are obviously independent of the chosen function $\bar{v}$.

These traces are called **one-sided approximate limits**.

### 2.2. Jump Conditions

In [3], we applied the following general jump condition to the initial-boundary value problem (2.1)–(2.4):

**Theorem 2.1** (Wu and Yin [17]). If $u$ is a generalized solution of Eq. (2.1) on $Q_T$, then the following conditions hold $H$-almost everywhere on $\Gamma_u$:

$$(u^+ - u^-) \nu_x + (f(u^+, t) - f(u^-, t)) \nu_x$$

$$- \left[ \left( a(u) \frac{\partial u}{\partial x} \right)^\prime \right] \nu_x = 0,$$  \hspace{1cm} (2.17)
\[ \forall u \in [u^*, u_*]: a(u) = 0, \quad (2.18) \]
\[ \forall k \in \mathbb{R}: \left[ \text{sgn}(u^* - k) - \text{sgn}(u^- - k) \right] \times \left[ (\bar{u} - k)v_* + \left( f(u, t) - f(k, t) \right)v_* - \left( a(u) \frac{\partial u}{\partial x} \right)v_* \right] \leq 0. \quad (2.19) \]

This theorem will be needed for the stability proof in Section 5.

2.3. Entropy Boundary Condition

The entropy boundary condition obtained in [3] enters into the stability proof as well. It may be formulated in the following way:

**Theorem 2.2.** Condition (2.14) in the definition of a generalized solution is satisfied if and only if the integral equality

\[ \int_Q \left( |u - k| \frac{\partial \varphi}{\partial t} + \text{sgn}(u - k) \times \left\{ f(u, t) - f(k, t) - a(u) \left( \frac{\partial u}{\partial x} \right) \frac{\partial \varphi}{\partial x} \right\} \right) dx dt \geq 0, \]

the condition \( a(\tau) = 0 \) for all \( \tau \in (\bar{u}(t), (\gamma u)(1, t)) \) and the following entropy boundary inequality are satisfied: For almost all \( k \in \mathbb{R} \), there holds almost everywhere on \( [0, T] \)

\[ \left[ \text{sgn}(\gamma u - k) - \text{sgn}(\bar{u}_x(t) - k) \right] \times \left[ f(\gamma u, t) - f(k, t) - \gamma \left( a(u) \frac{\partial u}{\partial x} \right) \right] \|_{t=1} \geq 0. \quad (2.20) \]

3. SOLVABILITY OF THE REGULARIZED PROBLEM

3.1. The Regularized Problem

To prove existence of generalized solutions of problem (2.1)–(2.4), we consider the regularized quasilinear parabolic initial-boundary value problem

\[ \frac{\partial u_\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u_\varepsilon, t) = \frac{\partial}{\partial x} \left( a(u_\varepsilon) + \varepsilon \frac{\partial u_\varepsilon}{\partial x} \right), \quad \varepsilon > 0, \quad (3.1) \]
\[ (x, t) \in Q_T = (0, 1) \times (0, T), \]
\[ u_\varepsilon(x, 0) = u_\varepsilon^0(x), \quad x \in [0, 1], \quad (3.2) \]
\[-(a(u_\varepsilon) + \varepsilon) \frac{\partial u_\varepsilon}{\partial x} + f_{b_k}(u_\varepsilon) \bigg|_{x=0} = 0, \quad t \in [0, T], \]  
\[u_\varepsilon(1,t) = \overline{u}_1^\varepsilon(t), \quad t \in [0, T] \]  
and show that the limit \( \varepsilon \to 0 \) of its solutions exists. By (3.2), the initial condition is approximated as well, as the modification of the diffusion coefficient affects the first order compatibility conditions at \( x = 0, t = 0 \) and \( x = 1, t = 1 \). Instead of (2.10)--(2.12), we now require

\[u_\varepsilon^0(1) = \overline{u}_1(0), \]  
\[-q(0)(u_\varepsilon^0)'(1) - f_{b_k}(u_\varepsilon^0(1))(u_\varepsilon^0)'(1) \]
\[-\left[a'(u_\varepsilon^0(1))(u_\varepsilon^0)'(1))^2 + (a(u_\varepsilon^0(1)) + \varepsilon)(u_\varepsilon^0)''(1) \right] = \overline{u}_1(0), \]  
\[a(u_\varepsilon^0(0)) + \varepsilon \right] (u_\varepsilon^0)'(0) - f_{b_k}(u_\varepsilon^0(0)) = 0, \]  

hence \( u_\varepsilon^0 \neq u_0 \) in general. However, a slight modification of the given initial function is sufficient to satisfy the new compatibility conditions (3.5)--(3.7). For example, if the data \( u_0(x) \) and \( \overline{u}_1(t) \) satisfy conditions (2.10)--(2.12), we can choose \( u_\varepsilon^0(x) := u_0(x) + h_\varepsilon(x) \) for \( x \in [0,1] \) with

\[
h_\varepsilon(x) := \begin{cases} 
-u_\varepsilon^0(0) \\
\frac{-u_\varepsilon^0(0)}{a(u_\varepsilon^0(0)) + \varepsilon} e^x (\varepsilon - x)^2, & x \in [0, \varepsilon], \\
0, & x \in (\varepsilon, 1 - \varepsilon), \\
\frac{-u_\varepsilon^0(1)}{a(u_\varepsilon^0(1)) + \varepsilon} e^x \\
\times (x - (1 - \varepsilon))^2 (1 - x)^3, & x \in [1 - \varepsilon, 1], 
\end{cases}
\]

such that conditions (3.5)--(3.7) are satisfied, and there hold \( h_\varepsilon \in C^2[0,1] \) and

\[
\max_{x \in [0,1]} |h_\varepsilon(x)| \leq \max \left\{ \frac{27}{256} \varepsilon |u_\varepsilon^0(0)|, \frac{243}{3125} \varepsilon^3 |u_\varepsilon^0(1)| \right\},
\]
\[
\int_0^1 |h_\varepsilon(x)| \, dx \leq \frac{27}{128} \varepsilon |u_\varepsilon^0(0)| + \frac{486}{3125} \varepsilon^2 |u_\varepsilon^0(1)|
\]

and

\[
\int_0^1 |h_\varepsilon''(x)| \, dx \leq 6 |u_\varepsilon^0(0)| + 15 \varepsilon |u_\varepsilon^0(1)|,
\]
where the rational constants are obtained by elementary discussion of local extrema. In addition, we will assume that
\[ \exists \varepsilon_0 > 0: \forall 0 < \varepsilon \leq \varepsilon_0: 0 \leq u_0^\varepsilon(x) \leq 1, x \in [0, 1] \]  
(3.11)
is valid and \( \varepsilon \leq \varepsilon_0 \), where the assumption \( 0 \leq u_1(t) \leq 1 \) is maintained.

3.2. Existence of a Solution of the Regularized Problem

We consider the solvability of the regularized parabolic initial-boundary value problems in the Hölder spaces \( H^l(\Omega) \) and \( H^{1+\beta/2}(\Omega_T) \), where \( l \) is always a non-integer positive number and \( \Omega = (0, 1) \). Our notation is adopted from [10]. For the solvability of (3.1)–(3.4) in \( H^{2+\beta, 1+\beta/2}(\Omega_T) \subset C^{2,1}(\Omega_T) \) for any \( \beta \in (0, 1) \), we suppose that the initial and boundary data possess the regularity properties
\[ u_0^\varepsilon \in H^{2+\beta}([0, 1]) \quad \text{uniformly in } \varepsilon \]  
(3.12)
and
\[ \bar{u}_1 \in H^{1+\beta/2}([0, T]) \]  
(3.13)
and that they satisfy the first order compatibility conditions (3.5)–(3.7).

**Theorem 3.1.** If the initial and boundary data satisfy the smoothness and compatibility conditions (3.12)–(3.13) and (3.5)–(3.7), the regularized problem (3.1)–(3.4) has a unique smooth solution \( u_\varepsilon \in H^{2+\beta, 1+\beta/2} \subset C^{2,1}(\Omega_T) \).

In [10, Chap. V], existence and uniqueness results are derived for initial-boundary value problems of quasilinear parabolic equations with uniform boundary conditions (for our equation, of the form \( u(t, x) = \bar{u}(t) \) or \( b_i(t, u) (\partial u / \partial x) + \psi(t, u) \right|_{x=i} = 0, i = 0, 1 \)) by a priori estimates on \( \max_{x \in \Omega} |u| \) and \( \max_{x \in \Omega} |u| \) and by the application of the Leray–Schauder fixed point theorem from theorems on corresponding standardized initial-boundary value problems of linear equations. These statements can also be formulated for linear equations with boundary conditions of the type (3.3) and (3.4) in a straightforward manner and can be proved following the lines of [10, Chap. IV]. Problem (3.1)–(3.4) can be written with the homogeneous initial condition for \( \hat{u}(x, t) := u_\varepsilon(x, t) - u_0^\varepsilon(x) \) and setting
\[ a^\varepsilon(x, \hat{u}) := a(\hat{u} + u_0^\varepsilon(x)) + \varepsilon, \]
\[ b(x, t, \hat{u}, \hat{u}_t) := \left[ f_{bb}(\hat{u} + u_0^\varepsilon) + q(t))(u_0^\varepsilon)' - (a(\hat{u} + u_0^\varepsilon) + \varepsilon)(u_0^\varepsilon)' \right] 
- a'(\hat{u} + u_0^\varepsilon)((u_0^\varepsilon)'')^2 
+ f_{bb}'(\hat{u} + u_0^\varepsilon) + q(t) - 2a'(\hat{u} + u_0^\varepsilon)(u_0^\varepsilon)' \right] \hat{u}_t 
+ \left[ -a'(\hat{u} + u_0^\varepsilon) \right] \hat{u}_t^2 \]
\[ \psi_1(t) := -\bar{u}_1(t) + u_0^\varepsilon(1), \]
\[ \psi_0(t, \hat{u}) := (a(\hat{u}(0, t) + u_0^\varepsilon(0) + \varepsilon)(u_0^\varepsilon)'(0) - f_{bb}(\hat{u}(0, t) + u_0^\varepsilon(0)) \]
assumes its minimum on $Q$. Let $u(x, t)$ solve the auxiliary problem

$$\mathcal{L}(x) u = a^\epsilon(x, \hat{u}) \hat{u}_{xx} + b(x, t, \hat{u}, \hat{u}_x) = 0 \quad \text{on } Q_T, \quad (3.14)$$

$$\mathcal{L}^{(S)}(x) u = -a^\epsilon(0, \hat{u}) \hat{u}_x + \psi_0(t, \hat{u}) = 0 \quad \text{on } S_0 T := \{0\} \times (0, T), \quad (3.15)$$

$$\mathcal{L}^{(S)}(x) u = \hat{u}(1, t) + \psi_1(t) = 0 \quad \text{on } S_{1T} := \{1\} \times (0, T). \quad (3.16)$$

Next it will be embedded into a parametric family of problems with parameter $\tau \in [0, 1]$. For $\tau = 0$ we obtain a heat conduction problem.

$$\mathcal{L}_\tau u = \tau \mathcal{L}_0 u + (1 - \tau) \mathcal{L}_0 u = 0 \quad \text{with } \mathcal{L}_0 u = u_t - \mu uu_x, \mu > 0, \quad (3.17)$$

$$\mathcal{L}^{(S)}_\tau u = \tau \mathcal{L}^{(S)}_0 u + (1 - \tau) \mathcal{L}^{(S)}_0 u = 0 \quad \text{with } \mathcal{L}^{(S)}_0 u = \mu(u_x - u), \quad (3.18)$$

$$\mathcal{L}^{(S)}_\tau u = \mathcal{L}^{(S)}_0 u = 0, \quad u|_{t=0} = 0. \quad (3.19)$$

Let $u^\tau(x, t)$ be the solution of (3.17)–(3.19) for a fixed value of $\tau \in [0, 1]$.

**Theorem 3.2.** If $u^\tau(x, t), (x, t) \in Q_T$ is a classical solution of the initial-boundary value problem (3.17)–(3.19) and if conditions (2.5), (2.6), and (3.11) are satisfied, then the following estimate holds for all $\tau \in [0, 1]$: \[ -u^\tau(x) \leq u^\tau(x, t) \leq 1 - u^\tau(x) \quad \text{for } (x, t) \in \overline{Q}_T. \quad (3.20) \]

In particular, there exists a constant $M$ with $\max_{\overline{Q}_T} |u^\tau(x, t)| \leq M$.

For $\tau = 1$, (3.20) reads as

$$0 \leq u_1(x, t) \leq 1 \quad \text{for } (x, t) \in \overline{Q}_T. \quad (3.21)$$

**Proof of Theorem 3.2.** Let $v(x, t) := e^{-\beta t} u^\tau(x, t)$ with $\beta > 0$. Then $v$ solves the auxiliary problem

$$v_t + \beta v + \tau((q(t) + f_{\beta k}(u^\tau + u^\delta))v_x - a^\epsilon(u^\tau + u^\delta)v_x^2) - \tau(a(u^\tau + u^\delta) + \epsilon) + (1 - \tau) u v_{xx} = 0 \quad (3.22)$$

$$v(t, x) = e^{-\beta t} u_1(t), \quad v(x, 0) = u^\delta(x). \quad (3.23)$$

We will show now that $v(x, t) \geq 0$ holds for $(x, t) \in \overline{Q}_T$. Suppose that $v$ assumes its minimum on $Q_T$ at the point $(x_0, t_0) \in \overline{Q}_T$.

1. $0 < x_0 < 1, 0 < t_0 \leq T$. In this case, $v_t \leq 0, v_x = 0$, and $v_{xx} \geq 0$
must be valid at \((x_0, t_0)\), hence
\[
\beta \cdot v(x_0, t_0) = \left( \frac{\tau (a u^\tau + u_0^\tau + e)}{1 - \tau} \right) v_x - v_i \geq 0,
\]
i.e., \(v(x_0, t_0) \geq 0\).

1. \(x_0 = 1, 0 < t_0 \leq T, v(x_0, t_0) = e^{-\beta \tau} \overline{u}(t_0) \geq 0\) (boundary condition at \(x = 1\)).

2. \(0 \leq x_0 \leq 1, t_0 = 0, v(x_0, t_0) = u_0^\tau(x_0) \geq 0\) (initial condition).

3. \(x_0 = 0, 0 < t_0 \leq T\). If the minimum is assumed at \((0, t_0)\) then \(v_i \geq 0\) must be valid. If \(v_i(x_0, t_0) = 0\), then \(v(x_0, t_0) \geq 0\) follows by the same argument as in case 1. For \(\tau = 1\), the assumption \(v_i(x_0, t_0) > 0\) leads to a contradiction, as it follows from boundary condition (3.23) and the nonpositivity of \(f_{vk}\) that
\[
v_i(0, t) = \frac{e^{-\beta \tau} f_{vk}(u^\tau + u_0^\tau)}{a(u^\tau + u_0^\tau) + e} \Bigg|_{t=0} \leq 0;
\]
for \(0 \leq \tau < 1\) boundary condition (3.23) yields for \(x = 0\)
\[
(1 - \tau) \mu (v(0, t_0) - e^{-\beta \tau} u_0^\tau(0))
\]
\[
= (\frac{\tau (a u^\tau + u_0^\tau + e)}{1 - \tau} \mu) v_x - (1 - \tau) \mu e^{-\beta \tau} (u_0^\tau)'
\]
\[
\geq 0
\]
\[
- \tau (1 - \tau) \mu e^{-\beta \tau} (u_0^\tau)'(0) \Rightarrow v(0, t_0) \geq e^{-\beta \tau} (u_0^\tau(0) - (u_0^\tau)'(0)).
\]
As \(\beta > 0\) can be chosen arbitrarily large, it follows that \(v(0, t_0) \geq 0\).

Hence \(v(x, t) \geq 0\) on \(\overline{Q}_T\), i.e., \(u^\tau(x, t) \geq -u_0^\tau(x)\) on \(\overline{Q}_T\). The initial-boundary value problem for \(v(x, t)\) can be formulated equivalently in terms of
\[
w(x, t) := 1 - u^\tau(x, t) - u_0^\tau(x) \quad \text{and} \quad \overline{w}(x, t) := e^{-\beta \tau} w(x, t), \quad \beta > 0.
\]
For brevity, we will not write out the corresponding auxiliary problems here and simply note that the boundary condition for \(w\) at \(x = 0\) reads
\[
\tau\left( - (a(1 - w) + e) w_x - f_{bk}(1 - w) \right) + (1 - \tau) \mu (-w_x - (1 - w)) \Bigg|_{x=0} = 0. \tag{3.24}
\]
Suppose now that $\tilde{v}$ assumes its minimum at $(x_1, t_1) \in \Omega_T$. For the cases $0 < x_1 \leq 1$, $0 \leq t_1 < T$ and $x_1 = 0$, $t_1 = 0$, we obtain $\tilde{v}(x_1, t_1) \geq 0$ by repeating the arguments used above. For the case $x_1 = 0$, $0 < t_1 \leq T$, we obtain from (3.24) that $\tilde{v}_s(0, t_1) \geq 0$ is valid. If $\tilde{v}_s(0, t_1) = 0$, the equation for $w$ on $\Omega_T$ yields $\tilde{v}(0, t_1) \geq 0$. Otherwise, in the case $\tilde{v}_s(x_1, t_1) > 0 \Leftrightarrow w_s(x_1, t_1) > 0$, it follows from (3.24) that

$$\tau f_{bs}(1 - w) = (- \tau (a(1 - w) + \varepsilon) - (1 - \tau) \mu)w,$$

$$- (1 - \tau) (1 - w) \leq 0.$$

For $\tau > 0$, this ensures $f_{bs}(1 - w) < 0$, i.e., $w \geq 0$. For $\tau = 0$ it follows that $1 - w = - \mu w_s < 0$, i.e., $w > 1$. Hence we have shown that $w \geq 0$ holds on $\Omega_T$, or, equivalently, $u^*(x, t) + u_0(x) \leq 1$ and hence for $\tau = 1$

$$- u_0^*(x) \leq \hat{u}(x, t) \leq 1 - u_0^*(x) \Leftrightarrow 0 \leq u_s(x, t) \leq 1 \quad \text{for} \quad (x, t) \in \Omega_T.$$

Theorem 3.2 is proved. \[ \Box \]

For $\tau = 1$ and $\varepsilon \to 0$ we obtain

**Corollary 3.1.** The generalized solution of (2.1)–(2.4) satisfies $0 \leq u(x, t) \leq 1$ almost everywhere on $\Omega_T$.

This means that the generalized solution assumes almost everywhere values which are physically relevant as volumetric solid concentrations. Introducing

$$a_s(x, u, u_s) := (\tau a^s(x, u) + (1 - \tau) \mu)u_s$$

(3.25)

$$a(x, t, u, u_s) := \tau (b(x, t, u, u_s) + a^s(u + u_0^s)u_s(u_0^s)'(x))$$

(3.26)

$$\psi(t, u) = \tau \psi_0(t, u) + (1 - \tau) \mu u(x, t)\big|_{t=0},$$

(3.27)

problem (3.17)–(3.19) can be written as a parabolic equation with principal part in divergence form,

$$u_t - \frac{\partial}{\partial x} (a_s(x, u, u_s)) + a(x, t, u, u_s) = 0,$$

(3.28)

with initial condition

$$u(x, 0) = 0$$

(3.29)
and boundary conditions

\[ u(1, t) = -\psi_1(t), \quad (3.30) \]

\[ -a^\epsilon(x, u) \frac{\partial u}{\partial x} + \psi(t, u) \bigg|_{x=0} = 0. \quad (3.31) \]

For \( \nu := \min(\varepsilon, \mu) > 0 \) and \( \lambda := \max(\max_u a(u) + \varepsilon, \mu) > 0 \) we have then

\[ \nu \leq a^\epsilon(x, u) \leq \lambda \text{ for } x \in [0, 1] \text{ and } -u_0^\epsilon(x) \leq u(x, t) \leq 1 - u_0^\epsilon(x). \quad (3.32) \]

From the definition of \( a_1 \) and \( a \) in (3.25) and (3.26) it follows that there exist constants \( \mu_1 \) and \( \nu_1 \) such that the inequalities

\[ a_1(x, u, p) \geq \nu_1 p^2 - \mu_1, \quad (3.33) \]

\[ |a_1(x, u, p) - (1 + p) + |a(x, t, u, p)| \leq \mu_2(1 + p)^2 \quad (3.34) \]

are satisfied for \( (x, t) \in \overline{Q_T} \). Further a priori estimates for the solution of (3.28)–(3.31) can be derived following the classical theory of quasilinear parabolic equations developed in [9, 10]. There it is shown that solutions of Eq. (3.28) with boundary conditions of the same type at both boundaries \( x = 0 \) and \( x = 1 \) (either as in (3.30) or as in (3.31)) belong to the Hölder space \( H^\alpha, a/2(\overline{Q_T}) \), where \( \alpha > 0 \) depends only on \( \mu_1, \nu_1, \mu_2, \nu_2, \max_u a(u) \) and the initial and boundary conditions. Moreover, it is shown that an a priori estimate of \( \max_{\overline{Q_T}}|u_0| \) can be obtained and that \( u \in H^{\alpha, a/2}(\overline{Q_T}) \), where, similarly, \( a \) depends only on \( \max_{\overline{Q_T}}|u_0| \) and constants which are known a priori. These results can be modified in a straightforward manner [2] for boundary conditions of different types at \( x = 0 \) and \( x = 1 \), as is the case here. Then we obtain the following lemma:

**Lemma 3.1.** Let \( u(x, t) = \tilde{u}^\epsilon(x, t) \) be a solution of problem (3.17), (3.19). Then there exist constants \( M_1, c, \) and \( \delta > 0 \) that depend only on \( M, \nu, \) and \( \lambda \) such that the following estimates hold:

\[ \max_{\overline{Q_T}}|u_0| \leq M_1, \quad \langle u \rangle_{\overline{Q_T}}^{(1 + \delta)} \leq c. \quad (3.35) \]

Here, \( \langle \cdot \rangle_{\overline{Q_T}}^{(1 + \delta)} \) denotes the norm belonging to the Hölder space \( H^{1+\delta, (1+\delta)/2}(\overline{Q_T}) \). From (3.12) we conclude that \( a^\epsilon(x, u) \) and \( b(x, t, u, p) \) belong to \( H^\beta([0, 1]) \) with respect to \( x \) with an exponent \( 0 < \beta < 1 \). Now we apply the Leray–Schauder principle to (3.17)–(3.19) to obtain existence of \( u \).
Theorem 3.3. Problem (3.17)–(3.19) has exactly one solution \( u^\tau(x, t) \) from \( H^{2+\beta, 1+\beta/2}(\Omega_T) \) for every \( \tau \in [0, 1] \).

Proof of Theorem 3.3. Set \( \alpha := \min(\delta, \beta) \), \( \delta \) from (3.35) and

\[
B_\alpha := \{ w \in H^{1+a,(1+a)/2}(\Omega_T^0) : w|_{t=0} = 0 \}.
\]

Now consider functions \( w \in B_\alpha \) satisfying \( \max_{Q_T}|w| \leq M \) and \( \max_{Q_T}|w_t| \leq M_1 \). Every function \( w \) defines a family of linear initial-boundary value problems, which result from (3.17)–(3.19) by replacing every occurrence of \( \hat{u} \) and \( \hat{u}_x \) in the coefficients by \( w \) and \( w_t \), respectively, i.e., writing

\[
\mathcal{L}(w)u = u_t - a^\tau(x, w)u_{xx} + b(x, t, w, w_t) = 0 \quad \text{on } Q_T, \tag{3.36}
\]

\[
\mathcal{L}^{(S_2)}(w)u = -a^\tau(0, w)u_x + \psi_0(t, w) = 0 \quad \text{on } S_{0T}, \tag{3.37}
\]

\[
\mathcal{L}^{(S_1)}u = u(1, t) + \psi_1(t) = 0 \quad \text{on } S_{1T}. \tag{3.38}
\]

We now consider the parametrized family of linear problems

\[
\mathcal{L}(w)u = \tau \mathcal{L}(w)u + (1 - \tau) \mathcal{L}_0(u) = 0
\]

with \( \mathcal{L}_0u = u_t - \mu u_{xx}, \mu > 0 \) \( \tag{3.39} \)

\[
\mathcal{L}^{(S_2)}(w)u = \tau \mathcal{L}^{(S_2)}(w)u + (1 - \tau) \mathcal{L}_0^{(S_2)}u = 0
\]

with \( \mathcal{L}_0^{(S_2)}u = \mu(u_x - u) \) \( \tag{3.40} \)

\[
\mathcal{L}^{(S_1)}u = \mathcal{L}^{(S_1)}u = 0, \quad u|_{t=0} = 0. \tag{3.41}
\]

The next theorem states the solvability of (3.39)–(3.41) for a fixed value of \( \tau \):

Theorem 3.4. The linear problem

\[
\begin{align*}
  u_t - a^\tau(x, t)u_{xx} + b(x, t)u_x + c(x, t)u &= f(x, t), \\
  u(x, 0) &= \phi(x), \quad u(1, t) = \Phi_1(t), \quad b_2(t)u_x + b_2(t)u|_{x=0} = \Phi_2(t)
\end{align*}
\]

with \( |b_2(t)| \geq \delta > 0 \) has exactly one solution from \( H^{l+2, l+1}(\Omega_T) \) for \( a, b, c, f \in H^{l, l/2}(\Omega_T), \phi \in H^{l+2}[0, 1], \Phi_1 \in H^{l+2}[0, T], \Phi_2 \in H^{l+1/2}[0, T] \), where the compatibility conditions of order \( [l + 1/2] \) at \( x = 0 \) and of order \( [l/2] + 1 \) at \( x = 1 \) are assumed to be satisfied. For this solution the uniform estimate

\[
|u|_{Q_T}^{l+2} \leq c \left( |f|_{Q_T}^{l+2} + |\phi|_{[0, 1]}^{l+2} + |\Phi_1|_{S_{1T}}^{l+2} + |\Phi_2|_{S_{0T}}^{l+2} \right) \tag{3.42}
\]

is valid.
The proof of Theorem 3.4 is outlined in [2] and mimics the proof of Theorem 5.3 in [10, Chap. IV, Sect. 5]. Theorem 3.4 ensures that every \( w \in B_a \) defines a transformation

\[
B_a \ni w \mapsto v = \Phi(w; \tau) \in B_a, \quad v \text{ solution of } (3.14)-(3.16)
\]

whose fixed points are the solutions of (3.17)-(3.19). Applying the Leray–Schauder fixed point theorem (e.g., [5]), we obtain from Theorem 3.4 the following result [11].

**Theorem 3.5.** For \( \tau = 1 \), problem (3.17)-(3.19) has exactly one solution from \( B_a \).

Theorem 3.3 is proved as well now, whose statement for \( \tau = 1 \) in turn proves Theorem 3.1.

### 4. Existence of a Generalized Solution

We will first show that the family \( \{u_\varepsilon\}_{\varepsilon > 0} \) of solutions of the regularized problems (3.1)–(3.4) forms a relatively compact subset of \( L^2(\Omega) \), i.e., that there exists a sequence \( \varepsilon = \varepsilon_n \to 0 \) such that \( \{u_\varepsilon\} \) converges in \( L^2(\Omega) \) to a function \( u \) which is even bounded and \( u \in BV(\Omega) \). Next we verify that the limit \( u \) is a generalized solution of the initial-boundary value problem.

We shall now derive some estimates for the solutions of the regularized problem. The estimate in Theorem 3.2 is independent of \( \varepsilon \); in particular, it follows that

\[
\exists M_1 > 0 : |u_\varepsilon(x, t)| \leq M_1. \quad (4.43)
\]

In what follows, we will use continuous approximations of the sign and modulus functions given by

\[
\text{sgn}_\eta(\tau) := \begin{cases} 
-1, & \tau < -\eta \\
\tau/\eta, & -\eta \leq \tau \leq \eta \\
1, & \tau > \eta 
\end{cases}
\]

and

\[
|x|_\eta := \int_0^x \text{sgn}_\eta(s) \, ds, \text{ where } \eta > 0.
\]

Differentiating Eq. (3.1) with respect to \( x \), we obtain that \( w := \partial u_\varepsilon/\partial x \) satisfies

\[
\frac{\partial w}{\partial t} + \frac{\partial^2}{\partial x^2} f(u_\varepsilon, t) = \frac{\partial^2}{\partial x^2} \left( a(u_\varepsilon) + \varepsilon \right) w. \quad (4.44)
\]
Multiplying (4.44) by $\operatorname{sgn}_\eta(w)$, integrating the result over $Q_T$, and using integration by parts yields
\[
\iint_{Q_T} \operatorname{sgn}_\eta(w) \frac{\partial w}{\partial t} 
\ \, dx \, dt \\
= \iint_{Q_T} \operatorname{sgn}_\eta(w) \frac{\partial^2}{\partial x^2} \left( (a(u_e) + \varepsilon)w - f(u_e, t) \right) 
\ \, dx \, dt \\
= \left[ \iint_{Q_T} \operatorname{sgn}_\eta(w) \frac{\partial}{\partial x} \left( (a(u_e) + \varepsilon)w - f(u_e, t) \right) \right]_{t=0}^{1} 
\ \, dt \\
- \iint_{Q_T} \operatorname{sgn}_\eta'(w) \frac{\partial w}{\partial x} \frac{\partial}{\partial x} \left( (a(u_e) + \varepsilon)w - f(u_e, t) \right) 
\ \, dx \, dt \\
= \left[ \iint_{Q_T} \operatorname{sgn}_\eta(w) \frac{\partial}{\partial x} \left( (a(u_e) + \varepsilon)w - f(u_e, t) \right) \right]_{t=0}^{1} 
\ \, dx \, dt \\
+ \iint_{Q_T} \operatorname{sgn}_\eta'(w) \frac{\partial w}{\partial x} f_a(u_e, t)w \, dx \, dt \\
- \iint_{Q_T} \operatorname{sgn}_\eta'(w) \frac{\partial w}{\partial x} a'(u_e)w^2 \, dx \, dt \\
- \iint_{Q_T} \operatorname{sgn}_\eta'(w) (a(u_e) + \varepsilon) \left( \frac{\partial w}{\partial x} \right)^2 \, dx \, dt.
\]

By the nonnegativity of the last integral and the initial condition, we obtain
\[
\int_0^1 w(x, T) \, \eta \, dx \leq \int_0^1 \left( (u_\xi)'(x) \right) \eta \, dx \\
+ \left[ \iint_{Q_T} \operatorname{sgn}_\eta(w) \frac{\partial}{\partial x} \left( (a(u_e) + \varepsilon)w - f(u_e, t) \right) \right]_{t=0}^{1} 
\ \, dt \\
+ \iint_{Q_T} \operatorname{sgn}_\eta'(w) \frac{\partial w}{\partial x} f_a(u_e, t)w \, dx \, dt \\
- \iint_{Q_T} \operatorname{sgn}_\eta'(w) \frac{\partial w}{\partial x} a'(u_e)w^2 \, dx \, dt \\
- \iint_{Q_T} \operatorname{sgn}_\eta'(w) (a(u_e) + \varepsilon) \left( \frac{\partial w}{\partial x} \right)^2 \, dx \, dt.
\]

The last two integrals vanish for $\eta \to 0$, due to the following lemma.
Lemma 4.1 [12, p. 130 f.] (see [1]). For a function \( v \in C^1(\Omega) \) there holds

\[
\lim_{\eta \to 0} \int_{\{x \in \Omega : |v(x)| < \eta\}} |\nabla v| \, dx = 0.
\]

By Eq. (3.1), the second integral in the right-hand side of (4.45) equals for \( \eta \to 0 \)

\[
\int_0^T \sgn w \frac{\partial u}{\partial t} \bigg|_{t=0}^1 dt = \int_0^T \left\{ \sgn(w(x,t)) \frac{\partial u}{\partial t}(t) \right\} dt.
\]

From the boundary condition for \( w \) at \( x = 0 \) which reads

\[
(a(u_x) + \varepsilon) \cdot w - fh_k(u_x) \big|_{x=0} = 0 \iff w(0,t) = \frac{k_1 h(u_x)}{a(u_x) + \varepsilon}(0,t),
\]

we can infer that if \( 0 < u_x(0,t) < 1 \), then \( w(0,t) < 0 \) and

\[
-\sgn(w(0,t)) \frac{\partial u}{\partial t} = \frac{\partial u}{\partial t} \quad \text{at } x = 0;
\]

otherwise, if \( u_x(0,t) \in (0,1) \), then \( w(0,t) = 0 \), and if the latter holds on an entire time interval \( [\tau_1, \tau_2] \), then \( (\partial u_x/\partial t)(0,t) = 0 \) for \( \tau_1 \leq t \leq \tau_2 \), as \( u_x \) is extremal there according to the estimate (3.21). Hence, we have

\[
\int_0^T (\sgn(w(0,t)) \frac{\partial u}{\partial t}(0,t) dt
\]

\[
= \int_0^T \frac{\partial u}{\partial t}(0,t) dt = u_x(0,T) - u_x(0,0),
\]

and we obtain with \( \eta \to 0 \) the estimate

\[
\int_0^T \left| \frac{\partial u_x}{\partial x} \right|(x,T) \, dx
\]

\[
\leq \int_0^T \left| \frac{\partial u_x}{\partial x} \right|(x,0) \, dx + \int_0^T |\overline{u}_1(t)| \, dt + u_x(0,T) - u_x(0,0)
\]

\[
= \int_0^T |(u_0')^\gamma(x)| \, dx + \int_0^T |\overline{u}_1(t)| \, dt + u_x(0,T) - u_x(0,0)
\]

\[
\leq \int_0^T |u_0'(x)| \, dx + \int_0^T |\overline{u}_1(t)| \, dt + u_x(0,T) - u_x(0,0) + \mathcal{E}(\varepsilon),
\]

(4.46)
i.e., there exists a constant $M_2$ independent of $\varepsilon$ such that

$$\int_0^1 \left| \frac{\partial u}{\partial x} \right| \, dx \leq M_2.$$  

To derive an estimate for $\partial u / \partial t$, we assume that on the time interval $[0, T]$, 

$$\overline{u}_t(t) \geq 0 \text{ on } [0, T] \quad \text{and} \quad \overline{u}_t(T) \geq 0 \quad (4.47)$$

is valid. The opposite case, $\overline{u}_t(t) < 0$ on $(0, T)$, can be treated analogously.

Differentiation of the viscous equation (3.1) with respect to $t$ yields for $v := \partial u / \partial t$ and $w = \partial u / \partial x$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial x} \left( (a(u_x) + \varepsilon)w - f_b(u_x) - q(t)u_x \right) \right]$$

$$= \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial t} \left( (a(u_x) + \varepsilon)w - f_b(u_x) - q(t)u_x \right) \right]. \quad (4.48)$$

Multiplying (4.48) by $(\text{sgn}_q(v) - \text{sgn}_q(\overline{u}_t(t)))$, we obtain

$$(\text{sgn}_q(v) - \text{sgn}_q(\overline{u}_t(t))) \frac{\partial v}{\partial t}$$

$$= \frac{\partial}{\partial x} \left[ (\text{sgn}_q(v) - \text{sgn}_q(\overline{u}_t(t))) \right]$$

$$\times \frac{\partial}{\partial t} \left( (a(u_x) + \varepsilon)w - f_b(u_x) - q(t)u_x \right)$$

$$- \text{sgn}_q'(v) \frac{\partial v}{\partial x} \frac{\partial}{\partial t} \left( (a(u_x) + \varepsilon)w - f_b(u_x) - q(t)u_x \right)$$

$$= \frac{\partial}{\partial x} \left[ (\text{sgn}_q(v) - \text{sgn}_q(\overline{u}_t(t))) \right]$$

$$\times \frac{\partial}{\partial t} \left( (a(u_x) + \varepsilon)w - f_b(u_x) - q(t)u_x \right)$$

$$- \text{sgn}_q'(v) (a(u_x) + \varepsilon) \left( \frac{\partial v}{\partial x} \right)^2$$

$$\leq 0$$

$$- \text{sgn}_q'(v) \frac{\partial v}{\partial x} (a'(u_x)w - f'_b(u_x) - q(t)v)$$

$$+ \text{sgn}_q'(v) \frac{\partial v}{\partial x} q'(t)u_x.$$
Integrating this inequality over \( Q \), \( 0 \leq T_0 \leq T \) leads to
\[
\int_{Q_{T_0}} (\text{sgn}_{\eta}(v) - \text{sgn}_{\eta}(\bar{\Pi}(t))) \frac{\partial v}{\partial t} \, dx \, dt 
\]
\[
\leq \int_0^{T_0} (\text{sgn}_{\eta}(v) - \text{sgn}_{\eta}(\bar{\Pi}(t))) \frac{\partial}{\partial x} \left( (a'(u) + \varepsilon)w - f_{\delta_k}(u_x) - q(t)u_x \right) \bigg|_0^1 dt 
\]
\[
- \int_0^{T_0} \text{sgn}_{\eta}(v) \, \frac{\partial v}{\partial x} \, (a'(u_x)w - f_{\delta_k}(u_x) - q(t)) \, dx \, dt 
\]
\[
+ \int_0^{T_0} \text{sgn}_{\eta}(v) \, q'(t)u_x \bigg|_0^1 dt - \int_0^{T_0} \text{sgn}_{\eta}(v) \, q'(t)w \, dx \, dt. 
\]

Due to the boundary conditions, there holds
\[
I_1 = \int_0^{T_0} (\text{sgn}_{\eta}(v) - \text{sgn}_{\eta}(\bar{\Pi}(t))) \left( (q'(t)u_x + q(t)v) \right) \bigg|_{t=0} \, dt. 
\]

By assumption (4.47), we obtain
\[
I_2 \xrightarrow{\eta \to 0} \int_0^{T_0} \text{sgn}(v) \, q'(t)u_x \bigg|_{t=0} \, dt + \int_0^{T_0} \frac{\partial q(t)}{\partial t}u_x \bigg|_{t=0} \, dt - \int_0^{T_0} \frac{\partial}{\partial t} \left( q(t)u_x \right) \, dt 
\]
\[
\leq \int_0^{T_0} \text{sgn}(v) \, q'(t)u_x \bigg|_{t=0} \, dt - q(t)u_x(0,t) \bigg|_0^{T_0} 
\]
\[
\leq T_0 \max_{0 \leq t \leq T_0} |q'(t)| + 2 \max_{0 \leq t \leq T_0} |q(t)| =: M_{33}. 
\]

The integrand in \( I_2 \) contains the factor \( \text{sgn}_{\eta}(v) \cdot v \). We conclude with Lemma 4.1 that \( I_2 \to 0 \) for \( \eta \to 0 \). Furthermore,
\[
I_3 \leq 2T_0 \max_{0 \leq t \leq T_0} |q'(t)| =: M_{33} 
\]
and
\[
I_4 \leq \int_{Q_{T_0}} \max_{0 \leq t \leq T_0} \left| \text{sgn}_{\eta}(v) \right| \max_{0 \leq t \leq T_0} \left| q'(t) \right| |w| \, dx \, dt 
\]
\[
\leq \max_{0 \leq t \leq T_0} |q'(t)| \int_0^{T_0} \left| \frac{\partial u_x}{\partial x} \right| \, dx \, dt \leq T_0 M_2 \max_{0 \leq t \leq T_0} |q'(t)| =: M_{34}. 
\]
Hence, there exists a constant \( M_{35} > 0 \) independent of \( \varepsilon \) such that

\[
\lim_{\eta \to 0} \int_{Q_T} \left( \text{sgn}_\eta(v) - \text{sgn}_\eta(\overline{u}_1(t)) \right) \frac{\partial v}{\partial t} \, dx \, dt \leq M_{31} + M_{33} + M_{34} =: M_{35}.
\]

(4.50)

On the other hand, we have

\[
\lim_{\eta \to 0} \int_{Q_T} \left( \text{sgn}_\eta(v) - \text{sgn}_\eta(\overline{u}_1(t)) \right) \frac{\partial v}{\partial t} \, dx \, dt
\]

\[
= \lim_{\eta \to 0} \int_{Q_T} \frac{\partial}{\partial t} |v|_\eta \, dx \, dt - \lim_{\eta \to 0} \int_0^1 \text{sgn}_\eta(\overline{u}_1(t)) v |_{0}^{T_0} \, dx
\]

\[
+ \lim_{\eta \to 0} \int_{Q_T} \text{sgn}_\eta(\overline{u}_1(t)) \overline{u}_1(t) \frac{\partial v}{\partial t} \, dx \, dt
\]

\[
= \int_0^1 |v|(x, T_0) \, dx - \int_0^1 |v|(x, 0) \, dx
\]

\[
- \text{sgn}(\overline{u}_1(T_0)) \int_0^1 v(x, T_0) \, dx - \text{sgn}(\overline{u}_1(0)) \int_0^1 v(x, 0) \, dx. \quad (4.51)
\]

Combining (4.50) and (4.51), we obtain

\[
\int_0^1 |v|(x, T_0) \, dx \leq M_{35} + \text{sgn}(\overline{u}_1(T_0)) \int_0^1 v(x, T_0) \, dx
\]

\[
+ \int_0^1 |v|(x, 0) \, dx + \text{sgn}(\overline{u}_1(0)) \int_0^1 v(x, 0) \, dx. \quad (4.52)
\]

By the initial condition \( u_\varepsilon(x, 0) = u_0^\varepsilon(x) \) and using Eq. (3.1), we have

\[
\int_0^1 |v|(x, T_0) \, dx \leq M_{35} + \text{sgn}(\overline{u}_1(T_0)) \int_0^1 v(x, T_0) \, dx
\]

\[
+ \int_0^1 \left[ \frac{d}{dx} \left( a(u_0^\varepsilon) + \varepsilon \right) (u_0^\varepsilon) - f_{b\varepsilon}(u_0^\varepsilon) - q(0)u_0^\varepsilon \right] \left[ \frac{d}{dx} \left( (a(u_0^\varepsilon) + \varepsilon) (u_0^\varepsilon)' - f_{b\varepsilon}(u_0^\varepsilon) - q(0)u_0^\varepsilon \right) \right] \, dx
\]

\[
\leq M_{35} + \text{sgn}(\overline{u}_1(T_0)) \int_0^1 v(x, T_0) \, dx. \quad (4.53)
\]
Integrating the last inequality over \(0 \leq T_0 \leq T\) yields
\[
\int_{Q_T} \left| \frac{\partial u_e}{\partial t} \right| \, dx \, dt \leq \int_{Q_T} \frac{\partial u_e}{\partial t} \, dx \, dt + T \cdot M_{36},
\]
from which we have
\[
\int_{Q_T} \left| \frac{\partial u_e}{\partial t} \right| \, dx \, dt \leq T \cdot M_{36} + \int_{Q_T} \frac{\partial u_e}{\partial t} \, dx \, dt
\]
\[
= T \cdot M_{36} + \int_0^1 (u_e(x, T) - u_e(x, 0)) \, dx
\]
\[
\leq T \cdot M_{36} + 1 =: M_{37}. \quad (4.54)
\]
Now suppose \(\bar{u}_T'(T) = 0, \bar{u}_T'(t) < 0\) for \(T \leq t \leq T'\). Inequality (4.53) then implies
\[
\int_0^1 |v(x, T)| \, dx \leq M_{36}. \quad (4.55)
\]
By repeating the arguments for the previous time interval, we obtain the following estimate similar to (4.52) for \(T \leq T_0 \leq T'\),
\[
\int_0^1 |v(x, T_0)| \, dx \leq \int_0^1 |v(x, T)| \, dx + \text{sgn}(\bar{u}_T'(T_0)) \int_0^1 v(x, T_0) \, dx + M_{35},
\]
(4.56)
that is, using (4.55),
\[
\int_0^1 |v(x, T_0)| \, dx \leq M_{36} + M_{35} + \text{sgn}(\bar{u}_T'(T_0)) \int_0^1 v(x, T_0) \, dx. \quad (4.57)
\]
Integrating inequality (4.57) over \([T, T']\), we obtain for \(Q_{T'} := (0, 1) \times (T, T')\)
\[
\int_{Q_{T'}} \left| \frac{\partial u_e}{\partial t} \right| \, dx \, dt
\]
\[
= \int_T^{T'} \left( \int_0^1 |v(x, t)| \, dx \right) \, dt
\]
\[
\leq (T' - T)(M_{35} + M_{36}) - \int_T^{T'} \left( \int_0^1 v(x, t) \, dx \right) \, dt
\]
\[
= (T' - T)(M_{35} + M_{36}) - \int_0^1 \left( \int_T^{T'} \frac{\partial u_e}{\partial t} (x, t) \, dt \right) \, dx
\]
\[
= (T' - T)(M_{35} + M_{36}) + \int_0^1 (u_e(x, T) - u_e(x, T')) \, dx
\]
\[
\leq (T' - T)(M_{35} + M_{36}) + 1 =: M_{37}. \quad (4.58)
\]
Continuing this reasoning over the monotonicity domains of $\bar{u}_1(t)$ we can conclude that there exists a constant $M > 0$ independent of $\varepsilon > 0$ such that

$$\int \int_{Q_T} \left| \frac{\partial u_\varepsilon}{\partial t} \right| dx dt \leq M.$$ 

Finally we multiply (3.1) by $(u_\varepsilon - \bar{u}_1(t))$ to obtain

$$(u_\varepsilon - \bar{u}_1(t)) \left[ \frac{\partial u_\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u_\varepsilon, t) \right] = \frac{\partial}{\partial x} \left( (u_\varepsilon - \bar{u}_1(t))(a(u_\varepsilon) + \varepsilon) \frac{\partial u_\varepsilon}{\partial x} \right) - (a(u_\varepsilon) + \varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x} \right)^2,$$

i.e.,

$$(a(u_\varepsilon) + \varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x} \right)^2 = -\frac{\partial}{\partial t} \left( \frac{1}{2} u_\varepsilon^2 - \bar{u}_1(t) u_\varepsilon \right) - u_\varepsilon \bar{u}_1'(t) + \frac{\partial u_\varepsilon}{\partial x} f(u_\varepsilon, t) - \frac{\partial}{\partial x} \left( (u_\varepsilon - \bar{u}_1(t)) \left[ f(u_\varepsilon, t) - (a(u_\varepsilon) + \varepsilon) \frac{\partial u_\varepsilon}{\partial x} \right] \right).$$

Integrating this over $Q_T$ and using the boundary conditions,

$$\int \int_{Q_T} (a(u_\varepsilon) + \varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x} \right)^2 dx dt = -\int_0^1 \left( \frac{1}{2} u_\varepsilon^2 - \bar{u}_1(t) u_\varepsilon \right) dx - \int \int_{Q_T} u_\varepsilon \bar{u}_1'(t) dx dt + \int \int_{Q_T} \frac{\partial u_\varepsilon}{\partial x} f(u_\varepsilon, t) dx dt - \int_0^T (u_\varepsilon - \bar{u}_1(t)) q(t) u_\varepsilon|_{t=0} dt.$$

Using the estimates derived before, we arrive at

$$\int \int_{Q_T} (a(u_\varepsilon) + \varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x} \right)^2 dx dt \leq \frac{3}{2} M_1^2 + T \max_{0 \leq t \leq T} |\bar{u}_1(t)| M_1 + \max_{0 \leq t \leq T} |f(u, t)| TM_1 + T \max_{0 \leq t \leq T} |q(t)| M_2^2 =: M_4.$$ 

Summing up, we have proved
**Theorem 4.1.** Let \( u_\varepsilon \in C^{2,1}(Q_T) \) be a solution of problem (3.1)-(3.4). Then there exist constants independent of \( \varepsilon \) such that the following uniform estimates hold:

\[
|u_\varepsilon(x,t)| \leq M_1, \quad \int_Q \left| \frac{\partial u_\varepsilon}{\partial t} \right| dx \leq M_3, \quad \int_0^1 \left| \frac{\partial u_\varepsilon}{\partial x} \right| dx \leq M_2,
\]

\[
\int_0^T \left( a(u_\varepsilon) + \varepsilon \right) \left( \frac{\partial u_\varepsilon}{\partial x} \right)^2 dx dt \leq M_4. \tag{4.59}
\]

The first three estimates establish the assumptions of Kolmogorov's compactness criterion (see, e.g., [15]). Hence we have

**Theorem 4.2.** The family \( \{u_\varepsilon\}_{\varepsilon > 0} \) of solutions of regularized problems (3.1)-(3.4) is compact in \( L^2(Q_T) \), i.e., there exists a sequence \( \varepsilon = \varepsilon_n \to 0 \) such that \( \{u_\varepsilon\} \) converges in \( L^2(Q_T) \) to a bounded function \( u \in BV(Q_T) \).

Before showing that the limit defines a generalized solution, we have to introduce some test functions. Let \( \delta \in C^\infty(\mathbb{R}) \) with \( \delta(\sigma) \geq 0, \delta(\sigma) = 0 \) for \( |\sigma| \geq 1, \int_{-\infty}^{\infty} \delta(\sigma) d\sigma = 1 \) and set for \( h > 0 \)

\[
\delta_h(\sigma) := \frac{1}{h} \delta\left( \frac{\sigma}{h} \right), \quad \phi_h(\sigma) := \int_{-\infty}^{\sigma} \delta_h(\tau) d\tau, \tag{4.60}
\]

\[
\mu_h(\sigma) := 1 - \phi_h(\sigma - 2h), \quad \nu_h(\sigma) := \phi_h(\sigma - (1 - 2h)). \tag{4.61}
\]

For these test functions, the following lemmata given in [16] are valid:

**Lemma 4.2.** Let \( v(x,t) \in L^1(Q_T) \) and \( |v(x,t)| \leq \Phi(t) \) a.e. in \( Q_T \) with a function \( \Phi(t) \in L^1(0,T) \). If \( (\gamma v)(0,t) \) and \( (\gamma v)(1,t) \) exist a.e. in \( [0,T] \), then we have for \( \varphi \in C^\infty(Q_T) \)

\[
\lim_{h \to 0} \int_Q \frac{\partial}{\partial x} (\varphi(x,t) \mu_h(x)) v(x,t) dx dt = -\int_0^T \varphi(0,t)(\gamma v)(0,t) dt,
\]

\[
\lim_{h \to 0} \int_Q \frac{\partial}{\partial x} (\varphi(x,t) \nu_h(x)) v(x,t) dx dt = \int_0^T \varphi(1,t)(\gamma v)(1,t) dt. \tag{4.62}
\]

**Lemma 4.3.** If \( v(x,t) \in L^1(Q_T) \) and \( \partial v/\partial x \) is an absolutely continuous measure, then for \( \varphi \in C^\infty(Q_T) \) with \( \text{supp} \varphi \subset [0,1] \times (0,T) \) there holds

\[
\int_Q \frac{\partial}{\partial x} \varphi v dt dx = \int_0^T \varphi(1,t)\gamma v_{\text{sc}} dt - \int_Q \varphi \frac{\partial v}{\partial x} dt dx. \tag{4.63}
\]

The uniform estimate (4.59) implies that \( \{\tau(u_\varepsilon)(\partial u_\varepsilon/\partial x)\}_{\varepsilon > 0} \) is weakly compact in \( L^2(Q_T) \), i.e., there exists a subsequence \( \{\varepsilon_n\} \) of \( \{\varepsilon\} \). Without
loss of generality we may take \( \{e_n\} \) itself, such that \( (r(u_{e_n}) \frac{\partial u_{e_n}}{\partial x}) \) converges weakly in \( L^2(Q_T) \) to a function \( g \). Hence for every \( \phi \in C^2(Q_T) \) there holds

\[
\int_{Q_T} \phi g \, dx \, dt = \lim_{e_n \to 0} \int_{Q_T} \phi r(u_{e_n}) \frac{\partial u_{e_n}}{\partial x} \, dx \, dt = \int_{Q_T} \phi r(u) \frac{\partial u}{\partial x} \, dx \, dt.
\]

Thus, condition (2.13) of the definition of generalized solutions is satisfied. To establish the integral inequality (2.14), we multiply the viscous equation (3.1) by \( \text{sgn}(u_e - k) \varphi \), where \( \varphi \in C^\infty(Q_T) \), \( \varphi \geq 0 \), \( \text{supp} \varphi \subset (0,1) \times (0,T) \), and \( k \in \mathbb{R} \). Integrating over \( Q_T \) and integration by parts yields

\[
\int_0^1 \left( |u_e - k|_\eta \varphi \big|_{t=0}^{+T} - \int_0^T |u_e - k|_\eta \frac{\partial \varphi}{\partial t} \, dt \right) \, dx
\]

\[
= 0 + \int_{Q_T} \text{sgn}_\eta(u_e - k) \varphi \frac{\partial}{\partial x} \left( f(u_e, t) - f(k, t) \right) \, dx \, dt
\]

\[
= \int_{Q_T} \text{sgn}_\eta(u_e - k) \varphi \frac{\partial}{\partial x} \left( (a(u_e) + e) \frac{\partial u_e}{\partial x} \right) \, dx \, dt.
\]

Using \( A(u) = \int_0^u a(\tau) \, d\tau \), we obtain

\[
- \int_{Q_T} |u_e - k|_\eta \frac{\partial \varphi}{\partial t} \, dx \, dt + \int_0^T \text{sgn}_\eta(u_e - k) \varphi \left( f(u_e, t) - f(k, t) \right) \bigg|_{t=1} \, dt
\]

\[
- \int_{Q_T} \text{sgn}_\eta(u_e - k) \left( f(u_e, t) - f(k, t) \right) \frac{\partial \varphi}{\partial x} \, dx \, dt
\]

\[
- \int_{Q_T} \text{sgn}_\eta(u_e - k) \frac{\partial u_e}{\partial x} \left( f(u_e, t) - f(k, t) \right) \varphi \, dx \, dt
\]

\[
\rightarrow 0 \text{ for } \eta \to 0
\]

\[
= \int_0^T \text{sgn}_\eta(u_e - k) \left( a(u_e) + e \right) \frac{\partial u_e}{\partial x} \bigg|_{t=1} \, dt
\]

\[
- \int_{Q_T} \text{sgn}_\eta(u_e - k) \frac{\partial}{\partial x} \left( A(u_e) + e u_e - A(k) \right) \frac{\partial \varphi}{\partial x} \, dx \, dt
\]

\[
- \int_{Q_T} \text{sgn}_\eta(u_e - k) \left( \frac{\partial u_e}{\partial x} \right)^2 \left( a(u_e) + e \right) \varphi \, dx \, dt
\]

\[
\leq 0
\]
≤ \int_0^T \operatorname{sgn}(u_x - k)(a(u_x) + \varepsilon) \left. \frac{\partial u_x}{\partial x} \varphi \right|_{x=1} dt \\
- \int_0^T \operatorname{sgn}(\bar{u}_x(t) - k)(A(u_x) + \varepsilon u_x - A(k)) \left. \frac{\partial \varphi}{\partial x} \right|_{x=1} dt \\
+ \int_{Q_T} \operatorname{sgn}(u_x - k)(A(u_x) + \varepsilon u_x - A(k)) \frac{\partial^2 \varphi}{\partial x^2} dx dt
\]

Taking the limit $\eta \to 0$, and in view of $\nu_h(1) = 1$ and $\nu_h(1) = 0$, one gets

\[
\int_{Q_T} |u_x - k| \frac{\partial \varphi}{\partial t} dx dt \\
- \int_0^T \operatorname{sgn}(\bar{u}_x(t) - k)(f(u_x, t) - f(k, t)) \varphi(x, t) \bigg|_{x=1} dt \\
+ \int_{Q_T} \operatorname{sgn}(u_x - k)(f(u_x, t) - f(k, t)) \frac{\partial \varphi}{\partial t} dx dt \\
+ \int_0^T \operatorname{sgn}(\bar{u}_x(t) - k)(A(u_x) + \varepsilon u_x - A(k)) \left. \frac{\partial \varphi}{\partial x} \right|_{x=1} dt \\
- \int_0^T \operatorname{sgn}(\bar{u}_x(t) - k)(A(u_x) + \varepsilon u_x - A(k)) \frac{\partial^2 \varphi}{\partial x^2} dx dt
\]

\[
= \int_{Q_T} \left\{ |u_x - k| \frac{\partial \varphi}{\partial t} + \operatorname{sgn}(u_x - k)(f(u_x, t) - f(k, t)) \frac{\partial \varphi}{\partial x} \right\} dx dt \\
+ \int_{Q_T} \operatorname{sgn}(u_x - k)(A(u_x) + \varepsilon u_x - A(k)) \frac{\partial^2 \varphi}{\partial x^2} dx dt \\
- \int_0^T \operatorname{sgn}(\bar{u}_x(t) - k) \left( f(u_x, t) - f(k, t) - (a(u_x) + \varepsilon) \frac{\partial u_x}{\partial x} \right) \\
\times \varphi(x, t) \bigg|_{x=1} dt \\
- \int_0^T \operatorname{sgn}(\bar{u}_x(t) - k)(A(u_x) + \varepsilon u_x - A(k)) \\
\times \left. \frac{\partial (\varphi(x, t) \nu_h(x))}{\partial x} \right|_{x=1} dt \geq 0.
\]
The limit for $\varepsilon \to 0$ exists due to Lemma 4.3,

$$\int_{\Omega_T} \left\{ |u - k| \frac{\partial \varphi}{\partial t} + \text{sgn}(u - k)(f(u, t) - f(k,t)) \frac{\partial \varphi}{\partial x} \right\} \, dx \, dt$$

$$+ \int_0^T \text{sgn}(\gamma u - k)(A(\gamma u) - A(k)) \left. \frac{\partial \varphi}{\partial x} \right|_{x=1} \, dt$$

$$- \int_{\Omega_T} \text{sgn}(u - k)a(u) \frac{\partial u}{\partial x} \frac{\partial \varphi}{\partial x} \, dx \, dt$$

$$- \int_{\Omega_T} \text{sgn}(\bar{u}_i(t) - k) \left( f(u,t) - f(k,t) - a(u) \frac{\partial u}{\partial x} \right) \times \frac{\partial (\varphi(x,t) \nu_k(x))}{\partial x} \, dx \, dt$$

$$- \int_{\Omega_T} \text{sgn}(\bar{u}_i(t) - k)(A(u) - A(k)) \frac{\partial^2 (\varphi(x,t) \nu_k(x))}{\partial x^2} \, dx \, dt \geq 0,$$

and for $h \to 0$ by Lemma 4.2,

$$\int_{\Omega_T} \left\{ |u - k| \frac{\partial \varphi}{\partial t} + \text{sgn}(u - k)(f(u, t) - f(k,t)) - a(u) \frac{\partial u}{\partial x} \right\} \, dx \, dt$$

$$+ \int_0^T \text{sgn}(\bar{u}_i(t) - k) \left( f(\gamma u, t) - f(k,t) - a(u) \frac{\partial u}{\partial x} \right) \varphi(1,t) \, dt \geq 0,$$

yielding integral inequality (2.14). Here we used (see [16]) that

$$\text{sgn}(u - k)(A(u) - A(k)) \in BV(\Omega_T),$$

and that

$$\frac{\partial}{\partial x} \left( \text{sgn}(u - k)(A(u) - A(k)) \right) = \text{sgn}(u - k) \frac{\partial}{\partial x} (A(u) - A(k))$$

is an absolutely continuous measure (cf. [14, p. 371 f.]). From boundary condition (3.3) it follows that for all $\psi \in C_0^\infty(0,T)$ there holds, using the
function \( \mu_h \) from (4.61),
\[
0 = \int_0^T \left[ (a(u_x) + \varepsilon) \frac{\partial u_x}{\partial x} - f_{b_h}(u_x) \right]_{x=0} \psi(t) \, dt 
\]
(4.65)
\[
= -\int_{Q_r} \frac{\partial}{\partial x} \left[ (a(u_x) + \varepsilon) \frac{\partial u_x}{\partial x} - f_{b_h}(u_x) \right] \psi(t) \mu_h(x) \, dx \, dt 
= -\int_{Q_r} \frac{\partial}{\partial x} \left[ (a(u_x) + \varepsilon) \frac{\partial u_x}{\partial x} - f_{b_h}(u_x) \right] \psi(t) \mu_h(x) \, dx 
= -\int_{Q_r} \left[ (a(u_x) + \varepsilon) \frac{\partial u_x}{\partial x} - f_{b_h}(u_x) \right] \psi(t) \mu_h(x) \, dx. 
\]

Note that
\[
\lim_{h \to 0} \int_{Q_r} \frac{\partial u_x}{\partial t} \psi(t) \mu_h(x) \, dx \, dt = -\lim_{h \to 0} \int_{Q_r} u_x(t) \psi'(t) \mu_h(x) \, dx \, dt = 0, 
\]
and that
\[
\left| \int_{Q_r} q(t) \frac{\partial u_x}{\partial x} \psi(t) \mu_h(x) \, dx \right| 
\leq \max_{0 \leq t \leq T} |q(t)| \cdot \max_{0 \leq t \leq T} |\psi(t)| \cdot \int_0^T \int_0^1 \left| \frac{\partial u_x}{\partial x} \right| \mu_h(x) \, dx 
\leq \max_{0 \leq t \leq T} |q(t)| \cdot \max_{0 \leq t \leq T} |\psi(t)| \cdot \max_{0 \leq t \leq T} \int_0^1 \left| \frac{\partial u_x}{\partial x} \right| \mu_h(x) \, dx. 
\]

The last integral vanishes for \( h \to 0 \). Furthermore, we have
\[
-\int_{Q_r} \left[ (a(u_x) + \varepsilon) \frac{\partial u_x}{\partial x} - f_{b_h}(u_x) \right] \psi(t) \mu_h'(x) \, dx \, dt 
= -\int_{Q_r} \left[ \frac{\partial}{\partial x} (A(u_x) + \varepsilon u_x) - f_{b_h}(u_x) \right] \psi(t) \mu_h'(x) \, dx 
= -\int_0^T \left[ A(u_x) + \varepsilon u_x \right] \psi'(t) \mu_h'(x)_{|x=0} \, dt 
\]
\[
+ \int_{Q_r} (A(u_x) + \varepsilon u_x) \psi(t) \mu_h''(x) \, dx \, dt
\]
\[ + \iint_{Q_T} f_{bh}(u_\varepsilon) \psi(t) \mu_\varepsilon(x) \, dx \, dt \]

\[ \varepsilon \to 0 \int_{Q_T} A(u) \psi(t) \mu_\varepsilon(x) \, dx \, dt + \iint_{Q_T} f_{bh}(u) \psi(t) \mu_\varepsilon(x) \, dx \, dt \]

\[ = - \iint_{Q_T} \left[ \frac{a(u)}{\partial x} - f_{bh}(u) \right] \psi(t) \mu_\varepsilon(x) \, dx \, dt \]

\[ \text{as} \ h \to 0 \int_0^T \gamma \left( \frac{a(u)}{\partial x} - f_{bh}(u) \right) \psi(t) \, dt. \]

Hence, condition (2.15) follows from (4.65) by taking the limit \( \varepsilon \to 0 \).

From

\[ \left| \frac{\partial u_\varepsilon}{\partial t}(x, 0) \right| \]

\[ = \int_0^1 \left| \frac{\partial}{\partial x} \left[ \left( f_{bh}(u_\varepsilon) + q(t) u_\varepsilon \right) + a(u_\varepsilon) + \varepsilon \frac{\partial u_\varepsilon}{\partial x} \right] \right| (x, 0) \, dx \]

\[ \leq \max \left( f_{bh}(u_0^\varepsilon(x)) + q(0) \int_0^1 (u_0^\varepsilon)'(x) \right) \, dx \]

\[ + \max \left( a'(u_0^\varepsilon(x)) \int_0^1 (u_0^\varepsilon)'(x) \right)^2 \]

\[ + \max \left( a(u_0^\varepsilon(x)) + \varepsilon \int_0^1 (u_0^\varepsilon)''(x) \right) \, dx \]

\[ \leq \max \left( f_{bh}(u) + q(0) \int_0^1 u_0'(x) \right) \, dx \]

\[ + 2 \max \left( a'(u) \int_0^1 (u_0'(x))^2 \right) \, dx \]

\[ + \max \left( a(u) + \varepsilon \left[ \int_0^1 u_0''(x) \right] \, dx + 15 |u_0'(0)| \right) + \Theta(\varepsilon) \leq M_5, \]

we may infer that the initial condition (2.16) is satisfied, since \( M_5 \) is a constant depending only on the initial function \( u_0 \) and its derivatives. Thus, the limit of the solutions of the viscous problems satisfies the definition of generalized solutions; and the existence of a generalized solution is shown.
5. Stability and Uniqueness of Generalized Solutions

Before stating the stability theorem, we note two lemmata which are needed in its proof.

**Lemma 5.1** [16]. Let $0 < \alpha \leq 1$ be a constant and $K_\alpha$ be the subclass of $BV(Q_T)$ with the additional properties that every $u \in K_\alpha$ is bounded and the measure $\mu_u = a^{1-\alpha}(u)$ is absolutely continuous on the set $E_c$ of all points where $u$ is approximately continuous and can be expressed by a bounded measurable density function $p(x,t)$,

$$\mu_u(E) = \int_E p \, dx \, dt,$$

where $E$ is any measurable subset of $E_c$. Now let $a^\alpha(u)$ be locally Lipschitz continuous and $u \in K_\alpha$ be a generalized solution of (2.1) to (2.4). Then for almost all $t \in H$,

$$\gamma(a(u) \frac{\partial u}{\partial x})(1,t) = 0.$$

Using Lemma 5.1, one can show the following result for two solutions:

**Lemma 5.2.** Let $a(u)(t) = 0$ for $t \in H$, $a^\alpha(u)$ locally Lipschitz continuous, and $u, v \in K_\alpha$ be generalized solutions of (2.1)–(2.4). Then for $\varphi \in C^0(Q_T \cup \{1\} \times H)$:

$$\lim_{h \to 0^+} \int_{Q_T} \frac{\partial}{\partial x} \left( \varphi(x,t) \nu_h(x) \right) \text{sgn}(u - v) \left( a(u) \frac{\partial u}{\partial x} - a(v) \frac{\partial v}{\partial x} \right) \, dx \, dt = 0.$$

**Theorem 5.1 (Stability of Generalized Solutions).** Let $a^\alpha(u)$ be locally Lipschitz continuous. If $u$ and $v$ are generalized solutions of (2.1) with the same boundary conditions (2.3), (2.4) and the initial conditions

$$(\gamma u)(x,0) = u_0(x), \quad (\gamma v)(x,0) = v_0(x) \ a.e. \ in \ [0,1], \quad (5.66)$$

then

$$\int_0^1 |u(x,t) - v(x,t)| \, dx \leq \int_0^1 |u_0(x) - v_0(x)| \, dx \quad a.e. \ on \ [0,T]. \quad (5.67)$$

**Corollary 5.1 (Uniqueness of Generalized Solutions).** Under the conditions of Theorem 5.1., the initial-boundary value problem (2.1)–(2.4) has at most one generalized solution.
Proof of Theorem 5.1. Modifying Theorem 2.1 of [14] (see also Section 4 of [17]), the following inequality holds for any nonnegative function \( \varphi \in C^0_0(Q_T \cup \{1\} \times P) \):

\[
\iint_{Q_T} \left[ |u - v| \frac{\partial \varphi}{\partial t} + \text{sgn}(u - v) \left( f(u, t) - f(v, t) \right) \frac{\partial \varphi}{\partial x} \right. \\
\left. - \left( a(u) \frac{\partial u}{\partial x} - a(v) \frac{\partial v}{\partial x} \right) \right] \, dx \, dt \geq 0. \tag{5.68}
\]

For sufficiently small \( h \), we choose the test function

\[
\varphi(x, t) = (1 - \mu_h(x)) \psi_0(t) + (1 - \mu_h(x) - \nu_h(x)) \psi_1(t),
\]

where \( \psi_0, \psi_1 \in C^0_0(0, T) \), \( \psi_0, \psi_1 \geq 0 \), \( \text{supp} \psi_0 \subset P \) and \( \text{supp} \psi_1 \subset H \), obtaining from (5.68)

\[
\iint_{Q_T} \left[ |u - v| \left[ (1 - \mu_h) \psi_0 + (1 - \mu_h - \nu_h) \psi_1 \right] \\
+ \text{sgn}(u - v) \left[ f(u, t) - f(v, t) - \left( a(u) \frac{\partial u}{\partial x} - a(v) \frac{\partial v}{\partial x} \right) \right] \\
\times \left[ - \mu_h(x)(\psi_0(t) + \psi_1(t)) - \nu_h(x) \psi_1(t) \right] \right] \, dx \, dt \geq 0. \tag{5.69}
\]

With Lemma 5.2 and Lemma 4.2, (5.69) implies for \( h \to 0 \)

\[
\iint_{Q_T} \left\{ |u - v|(\psi_0(t) + \psi_1(t)) \right\} \, dx \, dt \\
\geq \int_0^T \text{sgn}(\gamma u - \gamma v) f(\gamma u, t) - f(\gamma v, t) (1, t) \psi_1(t) \, dt \\
- \int_0^T \text{sgn}(\gamma u - \gamma v) \\
\times \left[ f(\gamma u, t) - a(u) \frac{\partial u}{\partial x} - \left( f(\gamma v, t) - a(v) \frac{\partial v}{\partial x} \right) \right](0, t) \\
\times (\psi_0(t) + \psi_1(t)) \, dt
\]
\begin{align*}
&= \int_0^T \sgn(\gamma u - \gamma v)(f(\gamma u, t) - f(\gamma v, t))(1, t) \psi_1(t) \, dt \\
&\quad - \int_0^T \sgn(\gamma u - \gamma v)(q(t)(\gamma u - \gamma v))(0, t)(\psi_0(t) + \psi_1(t)) \, dt \\
&= \int_0^T \sgn(\gamma u - \gamma v)(f(\gamma u, t) - f(\gamma v, t))(1, t) \psi_1(t) \, dt \\
&\quad - \int_0^T q(t) |\gamma u - \gamma v|(0, t)(\psi_0(t) + \psi_1(t)) \, dt.
\end{align*}

\begin{equation}
\geq 0 \text{ because of } q(t) \leq 0
\end{equation}

By Theorem 2.2 and Lemma 5.1, we have
\begin{align*}
&\left[ \sgn(\gamma u - k) - \sgn(\overline{\mu}_3(t) - k) \right] \left[ f(\gamma u, t) - f(k, t) \right] \geq 0 \\
&\left[ \sgn(\gamma v - k) - \sgn(\overline{\mu}_3(t) - k) \right] \left[ f(\gamma v, t) - f(k, t) \right] \geq 0
\end{align*}
\begin{equation}
a.e. \text{ on } H \text{ f.a.a. } k \in \mathbb{R}.
\end{equation}

We see from (5.71) that these relations actually hold for all \( k \in \mathbb{R} \). Now we choose for \( t \in H \)
\begin{align*}
k &= \begin{cases} 
\gamma u & \text{if } \gamma u \in I(\overline{\mu}_3(t), \gamma v), \\
\overline{\mu}_3(t) & \text{if } \overline{\mu}_3(t) \in I(\gamma u, \gamma v), \\
\gamma v & \text{if } \gamma v \in I(\overline{\mu}_3(t), \gamma u),
\end{cases}
\end{align*}

to obtain
\begin{align*}
\sgn(\gamma u - \gamma v)[f(\gamma u, t) - f(\gamma v, t)]|_{k=1} \\
&= \sgn(\gamma u - \gamma v)[(f(\gamma u, t) - f(k, t)) - (f(\gamma v, t) - f(k, t))]|_{k=1} \\
&= \sgn(\gamma u - \gamma v)[f(\gamma u, t) - f(k, t)]|_{k=1} \\
&\quad - \sgn(\gamma u - \gamma v)[f(\gamma v, t) - f(k, t)]|_{k=1} \\
&= \sgn(\gamma u - k)[f(\gamma u, t) - f(k, t)]|_{k=1} \\
&\quad + \sgn(\gamma v - k)[f(\gamma v, t) - f(k, t)]|_{k=1}.
\end{align*}

By (5.71), the right-hand side of (5.72) is nonnegative, i.e., there follows
\begin{equation}
\int_{Q_T} |u - \psi_0(t) + \psi_1(t)| \, dx \, dt \geq 0.
\end{equation}

Now assume that \( 0 < t_1 < \cdots < t_N < T \) are the points where the test function \( \psi_0 + \psi_1 \) must necessarily vanish due to the disjoint supports of
their summands. Then (5.73) is equivalent to

\[ \int_0^1 |u - v| \psi' \, dx \, dt \geq 0 \]

\[ \forall \psi : \psi \in C_0^\infty(0, T), \psi \geq 0, \psi(t_1) = \cdots = \psi(t_N) = 0. \]

(5.74)

Let \( 0 < s_1 < \tau_1 < t_1 \) and choose in (5.74) \( \psi(t) = \varrho_h(t - s_1) - \varrho_h(t - \tau_1) \) with \( \varrho_h \) from (4.60). Then inequality (5.73) reads for sufficiently small \( h \)

\[ \int_0^1 (\delta_h(t - s_1) - \delta_h(t - \tau_1)) \int_0^1 |u(x, t) - v(x, t)| \, dx \, dt \geq 0, \]

from which we obtain for \( h \to 0 \)

\[ \int_0^1 |u(x, \tau_1) - v(x, \tau_1)| \, dx \leq \int_0^1 |u(x, s_1) - v(x, s_1)| \, dx. \]

(5.75)

Now the transition to the interval \((t_1, t_2)\) must be established. Let \( X_j := [0, 1] \times \{t_j\}, j = 1, \ldots, N. \) By condition (2.17) in Theorem 2.1, \( u^+ = u^- \) and \( v^+ = v^- \) hold \( H\)-almost everywhere on \( X_j \); hence for \( w = u - v \) we have \( w^+ = w^- \) \( H\)-almost everywhere on \( X_1 \) as well, i.e.,

\[ w(x, t_1) = \lim_{\delta \to 0} w(x, t_1 \pm \delta) \quad H \text{- a.e. on } X_1. \]

Thus,

\[ \int_0^1 |u(x, t_1) - v(x, t_1)| \, dx = \lim_{\delta \to 0} \int_0^1 |u(x, t_1 \pm \delta) - v(x, t_1 \pm \delta)| \, dx. \]

On \((t_1, t_2)\), there analogously holds for \( t_1 < s_2 < \tau_2 < t_2 \):

\[ \int_0^1 |u(x, \tau_2) - v(x, \tau_2)| \, dx \leq \int_0^1 |u(x, s_2) - v(x, s_2)| \, dx. \]

Consequently, (5.75) is valid even for \( 0 < s_1 < \tau_1 < t_2 \). Repeating the arguments developed above for \( X_2 \) to \( X_N \) yields the assertion of Theorem 5.1. Setting \( u_0 = v_0 \) yields uniqueness as proposed in Corollary 5.1.00

Remark. It was shown that the boundary condition at \( x = 0 \) formulated in Section 2, (2.3), is assumed pointwise by the generalized solution for almost all \( t \in [0, T] \), such that the discharge can in fact be controlled by the choice of \( \varrho(t) \). While the entropy boundary condition at \( x = 1 \) is in general valid for smooth flux density functions, the stability proof makes in
(5.70) explicit use of the assumption $\dot{q}(t) \leq 0 \Leftrightarrow q(t) \leq 0$, which corresponds to an outflow condition at $x = 0$. This condition is essential for the stability and the uniqueness of the generalized solution, as we can easily give an example for $q > 0$ where a generalized solution satisfying Definition 2.1 fails to be unique. To this end, consider $a(u) = 0$, $f_{ps}(u) = -u(1 - u)^2$, $q(t) = q = 0.1$, $u_0 = 0$, and $\bar{u} = 0$. Clearly $u = 0$ is a generalized solution of the corresponding initial-boundary value problem, but so is

$$u_2(x, t) = \begin{cases} 
0, & x > qt \\
1, & x < qt,
\end{cases}$$

as the jump satisfies the conditions postulated by Theorem 2.1. The choice $q > 0$ would correspond to the inflow of suspension of generally unknown concentration at $x = 0$.

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